第四章　抽樣與抽樣分配（考古題）

2006 年 10 月 22 日 最後修改

4.1（95-淡江-保險）

4. Let X_1, X_2, X_3 be three independent random variables each with mean 0 and variance 1. If letting $Y = 2X_1 - X_3$ and $Z = 2X_2 + X_3$, then what will be the correlation coefficient of these two random variables Y and Z?

(1) 0　(2) $-\frac{1}{5}$　(3) -1　(4) $\frac{3}{5}$　(5) $\frac{1}{5}$

【解】

$$\begin{aligned}\operatorname{cov}(Y,Z) &= \operatorname{cov}(2X_1 - X_3, 2X_2 + X_3)\\ &= \operatorname{cov}(2X_1, 2X_2) + \operatorname{cov}(2X_1, X_3) + \operatorname{cov}(-X_3, 2X_2) + \operatorname{cov}(-X_3, X_3)\\ &= \operatorname{cov}(X_3, X_3) = 1\end{aligned}$$

$$\operatorname{var}(Y) = \operatorname{var}(2X_1 - X_3) = 4\operatorname{var}(X_1) + \operatorname{var}(X_3) = 5$$

$$\operatorname{var}(Z) = \operatorname{var}(2X_2 + X_3) = 4\operatorname{var}(X_2) + \operatorname{var}(X_3) = 5$$

$$\rho_{Y,Z} = \frac{\operatorname{cov}(Y,Z)}{\sqrt{\operatorname{var}(Y)}\sqrt{\operatorname{var}(Z)}} = \frac{1}{\sqrt{5}\sqrt{5}} = \frac{1}{5}$$

4.2（94-雲科大-企管）

七、某公司各項產品每月銷售量都呈現常態分配。A 產品的常態分配是 N(100, 300)，B 產品的常態分配是 N(80, 200)，C 產品的常態分配是 N(120, 400)，各項產品的銷售量彼此不會相互影響，亦即三者相互獨立，請問：

(1) 公司每月平均銷售量爲何？(7%)

(2) 公司每月銷售量的標準差爲何？(8%)

(3) 每月總銷售量低於 270 的機率爲何？(10%)

【解】

(1)

$$E(X_A + X_B + X_C) = E(X_A) + E(X_B) + E(X_C) = 100 + 80 + 120 = 300$$

(2)

$$\operatorname{var}(X_A + X_B + X_C) = \operatorname{var}(X_A) + \operatorname{var}(X_B) + \operatorname{var}(X_C) = 300 + 200 + 400 = 900$$

$$\sigma_{X_A + X_B + X_C} = \sqrt{\operatorname{var}(X_A + X_B + X_C)} = \sqrt{900} = 30$$

(3)

$Y=X_A+X_B+X_C$ 爲常態分配：$\mu_Y=300$、$\sigma_Y=30$

$$P(Y<270)=P\left(z<\frac{270-300}{30}=-1\right)=0.1587 \quad （查表）$$

或從 $P(-1\le z\le 1)=68\%$ 來推算。

4.3（94-雲科大-資管）

13. Round-off error has a uniform distribution on $[-0.5,+0.5]$ and round-off errors are independent. A sum of 50 numbers is calculated where each is of the form XXX.D, rounded to XXX before adding. What is the probability that the total round-off error exceeds five?
(A) 0.0071 (B) 0.0081 (C) 0.0091 (D) 0.0101 (E) 0.0111

【解】

$Y=X_1+X_2+\cdots+X_{50}$

X_i 爲在 $[-0.5,+0.5]$ 的均等分配，X_i、X_j，$i\ne j$ 互相獨立，則

$$E(X_i)=0,\quad \operatorname{var}(X_i)=\frac{(0.5-(-0.5))^2}{12}=\frac{1}{12}$$

Y 近似常態分配（大樣本）：$E(Y)=0,\quad \operatorname{var}(Y)=50\times\frac{1}{12}$

$$P(Y>5)=P\left(z>\frac{5}{\sqrt{50/12}}=2.45\right)=0.0071 \quad （查表）$$

4.4（94-逢甲-工工）

3. 隨機變數 X 與 Y 的聯合機率密度為

$$f(x,y)=\begin{cases} c, & x>0, y>0, x+y<1 \\ 0, & 其他 \end{cases}$$

(a) (8 分) 找出 c 值。
(b) (10 分) 若 Z = X + Y，完整找出 Z 的 p.d.f.。

【解】

(a)

$$\int_0^1\int_0^{1-x} c\,dy\,dx=\int_0^1 c(1-x)\,dx=c\frac{1}{2}=1 \quad\Rightarrow\quad c=2$$

(b)

$$F(Z \le z) = \int_0^z \int_0^{z-x} 2dydx = z^2, \quad 0 < z < 1$$

$$f(z) = \frac{d}{dz}F(Z) = 2z, \quad 0 < z < 1$$

4.5（94-淡江-產經）

5. (五分) Let $\bar{X}$ be the mean of a random sample of size 25 from a normal distribution with mean=75 and variance=100. What is the distribution of $\bar{X}$?

【解】

$\bar{X}$ 爲 $\mu_{\bar{X}} = \mu = 75$ 、 $\sigma_{\bar{X}} = \sqrt{\sigma^2/n} = \sqrt{100/25} = 2$ 的常態分配。

4.6（94-淡江-財金）

2. Let X and Y be random variables, let μ_X (μ_Y) and σ_X^2 (σ_Y^2) be the mean and variance of X (Y), and let σ_{XY} be the covariance between X and Y.

A. (10%) Show that $E(Y^2) = \sigma_Y^2 + \mu_Y^2$.

B. (10%) Show that $E(XY) = \sigma_{XY} + \mu_X\mu_Y$.

C. (15%) Show that $\sigma_{XY}^2 \le \sigma_X^2\sigma_Y^2$. [Hint: One possible way to prove this needs to use $V(aX+bY) = a^2\sigma_X^2 + 2ab\sigma_{XY} + b^2\sigma_Y^2$.]

【解】

(a)

$$\sigma_Y^2 = E\left[(Y-\mu_Y)^2\right] = E\left[Y^2 - 2\mu_Y Y + \mu_Y^2\right] = E(Y^2) - 2\mu_Y E(Y) + \mu_Y^2 = E(Y^2) - \mu_Y^2$$

$$\Rightarrow \quad E(Y^2) = \sigma_Y^2 + \mu_Y^2$$

(b)

$$\sigma_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right] = E\left[XY - \mu_Y X - \mu_X Y + \mu_X\mu_Y\right]$$
$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y = E(XY) - \mu_X\mu_Y$$

$$\Rightarrow \quad E(XY) = \sigma_{XY} + \mu_X\mu_Y$$

(c)

$$\operatorname{var}(aX+bY) = a^2\sigma_X^2 + 2ab\sigma_{XY} + b^2\sigma_Y^2 \ge 0 \quad \Rightarrow \quad -2ab\sigma_{XY} \le a^2\sigma_X^2 + b^2\sigma_Y^2$$

令 $a = \sigma_Y, b = -\sigma_X \quad \Rightarrow \quad 2\sigma_Y\sigma_X\sigma_{XY} \le \sigma_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2 = 2\sigma_X^2\sigma_Y^2 \quad \Rightarrow \quad \sigma_X\sigma_Y\sigma_{XY} \le \sigma_X^2\sigma_Y^2$

令 $a = \sigma_Y, b = \sigma_X \quad \Rightarrow \quad -\sigma_X\sigma_Y\sigma_{XY} \le \sigma_X^2\sigma_Y^2$

因此 $(\sigma_X\sigma_Y\sigma_{XY})^2 \le (\sigma_X^2\sigma_Y^2)^2 \quad \Rightarrow \quad (\sigma_{XY})^2 \le \sigma_X^2\sigma_Y^2$

4.7（94-淡江-保險）

二、設隨機變數 X,Y 之聯合機率密度函數為 f(x,y) = 8xy , 0 < x < y <1 ，試求： （1） X 與 Y 之邊際機率密度函數。（8 分） （2） X+Y 之變異數：V(X+Y)。（6 分） （3） X 與 Y 之相關係數：ρ(X,Y)。（6 分）

【解】

(1)

$$f(x)=\int_x^1 f(x,y)\,dy=\int_x^1 8xydy=4x\left(1-x^2\right)$$

$$f(y)=\int_0^y f(x,y)\,dx=\int_0^y 8xydx=4yy^2=4y^3$$

(2)

$$\operatorname{var}(X+Y)=\operatorname{var}(X)+\operatorname{var}(Y)+2\operatorname{cov}(X,Y)$$

$$E(X)=\int_0^1 xf(x)\,dx=\int_0^1 4x^2\left(1-x^2\right)dx=\frac{4}{3}-\frac{4}{5}=\frac{8}{15}$$

$$E\left(X^2\right)=\int_0^1 x^2 f(x)\,dx=\int_0^1 4x^3\left(1-x^2\right)dx=\frac{4}{4}-\frac{4}{6}=\frac{1}{3}$$

$$\operatorname{var}(X)=E\left(X^2\right)-\left[E(X)\right]^2=\frac{1}{3}-\frac{8}{15}\times\frac{8}{15}=\frac{11}{225}$$

$$E(Y)=\int_0^1 yf(y)\,dy=\int_0^1 4y^4dy=\frac{4}{5}$$

$$E\left(Y^2\right)=\int_0^1 y^2 f(y)\,dy=\int_0^1 4y^5dy=\frac{4}{6}=\frac{2}{3}$$

$$\operatorname{var}(Y)=E\left(Y^2\right)-\left[E(Y)\right]^2=\frac{2}{3}-\frac{4}{5}\times\frac{4}{5}=\frac{6}{225}$$

$$E(XY)=\int_0^1\int_x^1 xyf(x,y)\,dydx=\int_0^1\int_x^1 8x^2y^2dydx=\int_0^1 8x^2\tfrac{1}{3}\left(1-x^3\right)dx=\frac{8}{3}\left[\frac{1}{3}-\frac{1}{6}\right]=\frac{4}{9}$$

$$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{4}{9}-\frac{8}{15}\times\frac{4}{5}=\frac{4}{225}$$

$$\operatorname{var}(X+Y)=\operatorname{var}(X)+\operatorname{var}(Y)+2\operatorname{cov}(X,Y)=\frac{11}{225}+\frac{6}{225}-2\times\frac{4}{225}=\frac{1}{25}$$

(3)

$$\rho=\frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)}\sqrt{\operatorname{var}(Y)}}=\frac{4/225}{\sqrt{11/225}\sqrt{6/225}}=\frac{4}{\sqrt{66}}$$

4.8（94-淡江-企管）

2. 淡江公司有 4 位員工,其薪資分別爲 2, 3, 3, 4 (萬元).

(1) 現自其中以不放回的方式隨機抽出兩位員工的薪資,試分別求出員工平均薪資 $\bar{X}$ 及薪資變異數 S^2 的抽樣分配. (10 分)

(2) 試分別求算 $\bar{X}$ 及 S^2 的期望值及變異數,同時驗證 $\bar{X}$ 及 S^2 是否分別爲 μ 及 σ^2 之不偏估計式. (10 分)

【解】

(1)

樣本空間		$\bar{X}$	S^2
X_1	X_1	2	0
X_1	X_2	2.5	0.5
X_1	X_3	2.5	0.5
X_1	X_4	3	2
X_2	X_1	2.5	0.5
X_2	X_2	3	0
X_2	X_3	3	0
X_2	X_4	3.5	0.5
X_3	X_1	2.5	0.5
X_3	X_2	3	0
X_3	X_3	3	0
X_3	X_4	3.5	0.5
X_4	X_1	3	2
X_4	X_2	3.5	0.5
X_4	X_3	3.5	0.5
X_4	X_4	4	0

$\Longrightarrow$

$\bar{X}$	$P(\bar{X})$
2	1/16
2.5	4/16
3	6/16
3.5	4/16
4	1/16

S^2	$P(S^2)$
0	6/16
0.5	8/16
2	2/16

(2)

$$\mu=\frac{2+3+3+4}{4}=3,\quad \sigma^2=\frac{2^2+3^2+3^2+4^2}{4}-3^2=0.5$$

$$E(\bar{X})=2\times\frac{1}{16}+2.5\times\frac{4}{16}+3\times\frac{6}{16}+3.5\times\frac{4}{16}+4\times\frac{1}{16}=3=\mu$$

$$E(S^2)=0\times\frac{6}{16}+0.5\times\frac{8}{16}+2\times\frac{2}{16}=0.5=\sigma^2$$

皆爲不偏估計量。

4.9（94-政大-財管）

4. (10 points) Transformation of random variables.

A. If $x\sim N(o,\sigma^2)$ and $y=2x^2$, then find out the probability density function of y. Is y a normal random variable?

B. Let $X\sim Beta(2,3)$, i.e X is a random variable form a beta distribution with parameters of 2, and 3 over the range [0,1]. What is the probability density function of $Y=X^2$?

【解】

(A)

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

$$y=2x^2 \quad\Rightarrow\quad x=\frac{1}{\sqrt{2}}y^{\frac{1}{2}} \quad\Rightarrow\quad J=\frac{dx}{dy}=\frac{1}{2\sqrt{2}}y^{-\frac{1}{2}}$$

$$f_Y(y)=|J|\,f_X(x)=\frac{1}{2\sqrt{2}}y^{-\frac{1}{2}}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}=\frac{1}{4\sqrt{\pi}\sigma}y^{-\frac{1}{2}}e^{-\frac{1}{4}\frac{y}{\sigma^2}},\quad y\ge 0$$

由 y 的範圍即可判斷不是常態分配。

(B)

$$f_X(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}=12x(1-x)^2,\quad 0\le x\le 1$$

$$y=x^2 \quad\Rightarrow\quad x=y^{\frac{1}{2}} \quad\Rightarrow\quad J=\frac{dx}{dy}=\frac{1}{2}y^{-\frac{1}{2}}$$

$$f_Y(y)=|J|\,f_X(x)=\frac{1}{2}y^{-\frac{1}{2}}12x(1-x)^2=6\left(1-\sqrt{y}\right)^2,\quad 0\le y\le 1$$

4.10（94-政大-風管）

2. Express $\operatorname{var}(2X+Y)$, $\operatorname{var}(X-3Y)$, and $\operatorname{cov}(3X+Y, X-3Y)$ in terms of the variances and covariance of X and Y. **(12 points)**

【解】

$$\operatorname{var}(2X+Y)=\operatorname{var}(2X)+\operatorname{var}(Y)+2\operatorname{cov}(2X,Y)=4\operatorname{var}(X)+\operatorname{var}(Y)+4\operatorname{cov}(X,Y)$$

$$\operatorname{var}(X-3Y)=\operatorname{var}(X)+\operatorname{var}(3Y)+2\operatorname{cov}(X,-3Y)=\operatorname{var}(X)+9\operatorname{var}(Y)-6\operatorname{cov}(X,Y)$$

$$\begin{aligned}\operatorname{cov}(3X+Y,X-3Y)&=\operatorname{cov}(3X,X)+\operatorname{cov}(3X,-3Y)+\operatorname{cov}(Y,X)+\operatorname{cov}(Y,-3Y)\\&=3\operatorname{var}(X)-9\operatorname{cov}(X,Y)+\operatorname{cov}(X,Y)-3\operatorname{var}(Y)\\&=3\operatorname{var}(X)-3\operatorname{var}(Y)-8\operatorname{cov}(X,Y)\end{aligned}$$

4.11（94-台大-商學）

2. A theorem that allows us to use the normal probability distribution to approximate the sampling distribution of sample means and sample proportions whenever the sample size is large is known as the A. Approximation Theorem B. Normal Probability Theorem C. Central Limit Theorem D. Central Normality Theorem

【解】

(C)

4.12（94-台大-商學）

3. As the number of degrees of freedom for a t distribution increases, the difference between the t distribution and the standard normal distribution

   A. Becomes Larger
   B. Becomes Smaller
   C. Stays the Same
   D. None of These Alternatives Is Correct

【解】

(B)

4.13（93-雲科大-資管）

12. Suppose that the length of life in hours, say X, of a light bulb manufactured by company A is normally distributed with mean 800 hours and variance 14400 $(\text{hours})^2$, and the length of life in hours, say Y, of a light bulb manufactured by company B is normally distributed with mean 850 hours and variance 2500 $(\text{hours})^2$. One bulb is selected from each company and burned until "death." What is the probability that the length of life of the bulb from company A exceeds the length of life of the bulb from company B by at least 15 hours?
(A) 0.2267 (B) 0.24 (C) 0.2673 (D) 0.2829 (E) 0.3085

【解】

X 爲常態分配：$\mu_X = 800$ 、$\sigma_X = \sqrt{14,400} = 120$

Y 爲常態分配：$\mu_Y = 850$ 、$\sigma_Y = \sqrt{2,500} = 50$

求 $P(X - Y \ge 15)$

$X - Y$ 爲常態分配：$\mu_{X-Y} = 800 - 850 = -50$ 、$\sigma_{X-Y} = \sqrt{14,400 + 2500} = 130$

$P(X - Y \ge 15) = P\left(z \ge \dfrac{15-(-50)}{130} = 0.5\right) = 0.3085$ （查表）

4.14（93-雲科大-資管）

14. Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from $N(0, \sigma^2)$. Find the constant C so that $C(X_1-X_2)/\sqrt{X_3^2+X_4^2+X_5^2}$ has a t-distribution.

(A) 0.5 (B) 0.6667 (C) 0.8165 (D) 1.2247 (E) 1.5

【解】

t 分配的形式爲 $\dfrac{z}{\sqrt{\chi^2/df}}$

$$t=\frac{\dfrac{X_1-X_2}{\sigma_{X_1-X_2}}}{\sqrt{\left[\left(\frac{X_3}{\sigma}\right)^2+\left(\frac{X_4}{\sigma}\right)^2+\left(\frac{X_5}{\sigma}\right)^2\right]\Big/3}}=\frac{X_1-X_2}{\sqrt{\sigma^2+\sigma^2}}\times\frac{\sqrt{3\sigma^2}}{\sqrt{X_3^2+X_4^2+X_5^2}}=\sqrt{\frac{3}{2}}\frac{X_1-X_2}{\sqrt{X_3^2+X_4^2+X_5^2}}$$

$$\Rightarrow\quad C=\sqrt{\frac{3}{2}}=1.2247$$

4.15（93-雲科大-資管）

15. The coefficient of variation (C.V.) for a sample of values $Y_1, Y_2, \ldots, Y_n$ is defined by

$$C.V. = S/\overline{Y}.$$

This quantity is sometimes informative. For example, the value $S=10$ has little meaning unless we can compare it to something else. If S is observed to be 10 and $\overline{Y}$ is observed to be 1000, the amount of variation is small relative to the size of the mean. However, if S is observed to be 10 and $\overline{Y}$ is observed to be 5, the variation is quite large relative to the size of the mean. Let $Y_1, Y_2, \ldots, Y_{10}$ denote a random sample of size ten from a normal distribution with mean 0 and variance σ^2. Find the number c such that

$$p(-c \le \frac{S}{\overline{Y}} \le c) = 0.95.$$

(A) 0.95 (B) 5.12 (C) 10 (D) 24.05 (E) 49.04

【解】

$\dfrac{\overline{Y}-\mu_{\overline{Y}}}{s_{\overline{Y}}}=\dfrac{\overline{Y}}{s/\sqrt{n}}$ 爲自由度 $n-1$ 的 t 分配

$t_{\frac{\alpha}{2}=0.05,df=9}=2.2622$ ，即

$$P\left(-c\le\frac{s}{\overline{Y}}\le c\right)=P\left(\frac{\overline{Y}}{s}\le-\frac{1}{c}\text{或}\frac{\overline{Y}}{s}\ge\frac{1}{c}\right)=P\left(\frac{\overline{Y}}{s/\sqrt{10}}\le-\frac{\sqrt{10}}{c}\text{或}\frac{\overline{Y}}{s/\sqrt{10}}\ge\frac{\sqrt{10}}{c}\right)=0.95$$

$$t^*_{\frac{\alpha}{2}=0.95,df=9} = 0.0645 = \frac{\sqrt{10}}{c} \quad \Rightarrow \quad c = \frac{\sqrt{10}}{0.0645} = 49.03$$

（自由度 9 的 t 分配，雙尾，$\alpha = 0.95$，查表得臨界值爲 0.0645。）

4.16（93-淡江-國貿）

一、(10%)請敘述中央極限定理(Central Limit Theorem)。

【解】

若 $X_1, X_2, \ldots, X_n$ 爲 i.i.d.，來自平均數 μ、標準差 σ 的母體，令

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

則當 $n \to \infty$ 時，$\overline{X}$ 會成爲平均數 μ、標準差 $\sigma/\sqrt{n}$ 的常態分配。

4.17（93-淡江-財金）

(3) Let $X_1, X_2, \ldots, X_n$ be independent random variables that all have the same probability distribution, with mean β and variance σ^2. Since we know that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then E[$\overline{X}$] and Var[$\overline{X}$] are the value of _____ and _____, respectively.

(a) β/n and σ^2/n (b) β and σ^2/n (c) β and σ^2 (d) β/n and σ^2

【解】

$$E\left(\overline{X}\right) = \beta, \quad \text{var}\left(\overline{X}\right) = \frac{\sigma^2}{n}$$

4.18（93-淡江-保險）

5. Let X and Y be two independent random variables with pdf's

$f_X(x) = e^{-x}$ $x > 0$, and $f_Y(y) = e^{-y}$ $y > 0$

Find $f_Z(z)$ for Z = X+Y.

【解】

$$f(z) = \int f_Y(z-x) f_X(x) dx = \int_0^z e^{-(z-x)} e^{-x} dx = ze^{-z}, \quad z > 0$$

4.19（93-淡江-保險）

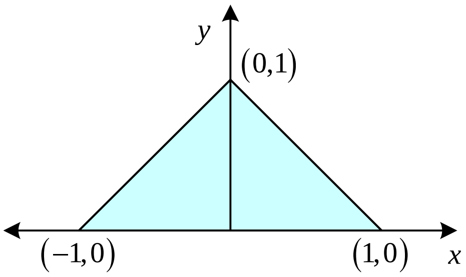
7. Suppose that X and Y are uniformly distributed over the triangle with vertices (1, 0), (0, 1) and (-1, 0).

(d) Find $\Pr(X \le 3/4, Y \le 3/4)$.

(e) Find $\Pr(X - Y \ge 0)$.

【解】

(x, y)範圍：

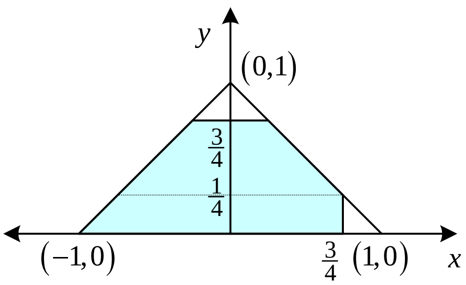


令 $f(x, y)=\dfrac{1}{m}$ ，則

$$\int_{y=0}^{1}\int_{x=y-1}^{1-y} f(x, y)\,dxdy=1 \quad \Rightarrow \quad \int_{y=0}^{1}\int_{x=y-1}^{1-y}\frac{1}{m}\,dxdy=\frac{1}{m}\int_{y=0}^{1}(2-2y)\,dy=\frac{1}{m}(2-1)=1$$

$$\Rightarrow \quad m=1$$

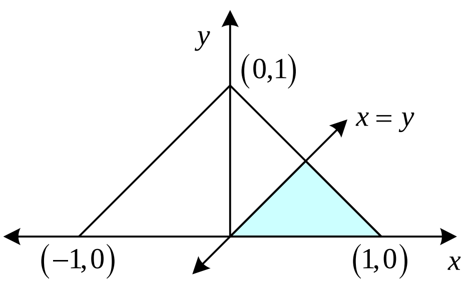
(d)



$$P\left(X \le \tfrac{3}{4}, Y \le \tfrac{3}{4}\right)=\int_{y=0}^{\frac{1}{4}}\int_{x=y-1}^{\frac{3}{4}} dxdy+\int_{y=\frac{1}{4}}^{\frac{3}{4}}\int_{x=y-1}^{1-y} dxdy=\int_{y=0}^{\frac{1}{4}}\left(\tfrac{7}{4}-y\right)dy+\int_{y=\frac{1}{4}}^{\frac{3}{4}}(2-2y)\,dy$$

$$=\left[\tfrac{7}{4}\times\tfrac{1}{4}-\tfrac{1}{2}\left(\tfrac{1}{4}\right)^2\right]+\left[2\times\left(\tfrac{3}{4}-\tfrac{1}{4}\right)-\left(\tfrac{3}{4}\right)^2+\left(\tfrac{1}{4}\right)^2\right]=\frac{29}{32}$$

(e)



$$P(X-Y \ge 0)=\int_{y=0}^{\frac{1}{2}}\int_{x=y}^{1-y} dxdy=\int_{y=0}^{\frac{1}{2}}(1-2y)\,dy=\tfrac{1}{2}-\left(\tfrac{1}{2}\right)^2=\frac{1}{4}$$

4.20（93-政大-財管）

6. (30%) Let Y_1, Y_2, Y_3, and Y_4 be independent, identically distributed random variables from a population with mean μ and variance σ^2.

Let $\bar{Y} = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)$ denote the average of these four random variables.

a. What are the expected value and variance of $\bar{Y}$ in terms of μ and σ^2?

b. Now, consider a different estimator of μ:

$$W = \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{4}Y_3 + \frac{1}{2}Y_4.$$

Show that W is also an unbiased estimator of μ and find the variance of W.

c. Based on your answers to parts (a) and (b), which estimator of μ do you prefer, $\bar{Y}$ or W?

【解】

(a)

$$E(\bar{Y}) = E\left[\tfrac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)\right] = \tfrac{1}{4}\left[E(Y_1) + E(Y_2) + E(Y_3) + E(Y_4)\right] = \tfrac{1}{4}(4\mu) = \mu$$

$$\begin{aligned}\operatorname{var}(\bar{Y}) &= \operatorname{var}\left[\tfrac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)\right] \\ &= \tfrac{1}{16}\left[\operatorname{var}(Y_1) + \operatorname{var}(Y_2) + \operatorname{var}(Y_3) + \operatorname{var}(Y_4)\right] = \tfrac{1}{16}(4\sigma^2) = \tfrac{1}{4}\sigma^2\end{aligned}$$

(b)

$$E(W) = E\left[\tfrac{1}{8}Y_1 + \tfrac{1}{8}Y_2 + \tfrac{1}{4}Y_3 + \tfrac{1}{2}Y_4\right] = \tfrac{1}{8}\mu + \tfrac{1}{8}\mu + \tfrac{1}{4}\mu + \tfrac{1}{2}\mu = \mu$$

$$\operatorname{var}(W) = \operatorname{var}\left[\tfrac{1}{8}Y_1 + \tfrac{1}{8}Y_2 + \tfrac{1}{4}Y_3 + \tfrac{1}{2}Y_4\right] = \left(\tfrac{1}{8}\right)^2\sigma^2 + \left(\tfrac{1}{8}\right)^2\sigma^2 + \left(\tfrac{1}{4}\right)^2\sigma^2 + \left(\tfrac{1}{2}\right)^2\sigma^2 = \tfrac{11}{32}\sigma^2$$

(c)

因 $\operatorname{var}(\bar{Y}) = \tfrac{1}{4}\sigma^2 < \operatorname{var}(W) = \tfrac{11}{32}\sigma^2$ ，所以就有效性（efficiency）而言，$\bar{Y}$ 比 W 好。

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11. Consider the following transformation where X_1 and X_2 are independent, each with the uniform distribution $U(0,1)$. Let

$z_1=\sqrt{-2\ln X_1}\cos(2\pi X_2)$, and $z_2=\sqrt{-2\ln X_1}\sin(2\pi X_2)$

Then, the joint p.d.f. of z_1 and z_2 is

(A) $g(z_1, z_2)=\dfrac{1}{\pi}\exp(-\dfrac{z_1^2+z_2^2}{2})$, $-\infty<z_1<\infty, -\infty<z_2<\infty$.

(B) $g(z_1, z_2)=\dfrac{1}{2\pi}\exp(-\dfrac{z_1^2+z_2^2}{2})$, $-\infty<z_1<\infty, -\infty<z_2<\infty$.

(C) $g(z_1, z_2)=\dfrac{1}{2\pi}\exp(\dfrac{z_1^2+z_2^2}{2})$, $-\infty<z_1<\infty, -\infty<z_2<\infty$.

(D) None of the above

【解】

(A)、(B)、(C)中只有(B)才是機率密度函數，因此只有(B)、(D)才可能是答案。

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(x^2+y^2\right)}dxdy=\int_0^{2\pi}\int_0^{\infty}re^{-\frac{1}{2}r^2}drd\theta=2\pi$$

$$\int_0^1\int_0^1 dx_1dx_2=1 \quad\Rightarrow\quad f\left(X_1,X_2\right)=1$$

$$\begin{cases} z_1=\sqrt{-2\ln X_1}\cos\left(2\pi X_2\right) \\ z_2=\sqrt{-2\ln X_1}\sin\left(2\pi X_2\right) \end{cases} \Rightarrow \begin{cases} X_1=e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} \\ X_2=\frac{1}{2\pi}\tan^{-1}\left(z_2/z_1\right) \end{cases}$$

$$\frac{\partial X_1}{\partial z_1}=-z_1e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)},\qquad \frac{\partial X_1}{\partial z_1}=-z_2e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}$$

$$\frac{\partial X_2}{\partial z_1}=-\frac{1}{2\pi}\frac{z_2}{z_1^2+z_2^2},\qquad \frac{\partial X_2}{\partial z_2}=\frac{1}{2\pi}\frac{z_1}{z_1^2+z_2^2}$$

$$J=\begin{vmatrix} \frac{\partial X_1}{\partial z_1} & \frac{\partial X_1}{\partial z_1} \\ \frac{\partial X_2}{\partial z_1} & \frac{\partial X_2}{\partial z_2} \end{vmatrix}=\begin{vmatrix} -z_1e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} & -z_2e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} \\ -\frac{1}{2\pi}\frac{z_2}{z_1^2+z_2^2} & \frac{1}{2\pi}\frac{z_1}{z_1^2+z_2^2} \end{vmatrix}=-\frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}$$

$$\Rightarrow\quad g\left(z_1,z_2\right)=\left|J\right|f\left(X_1,X_1\right)=\frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}$$