第三章　機率分配（考古題）

2006 年 10 月 23 日 最後修改

3.1（95-淡江-國貿）

2) A fair die is labelled with two faces showing a 1, two faces showing a 2, and two faces showing a 3. The die is tossed twice, and X_1 = the number on the top of the die on the first toss, X_2 = the number on the top of the die on the second toss.
a) Find $E[X_1]$ and $E[X_2]$. (6%)
b) Find the distribution of $Y = X_1 + X_2$, and compute $E[Y]$. (6%)
c) Are X_1 and X_2 independent? Why? (8%)

【解】

$P(X_1,X_2)$	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	邊際機率
$X_1 = 1$	1/9	1/9	1/9	1/3
$X_1 = 2$	1/9	1/9	1/9	1/3
$X_1 = 3$	1/9	1/9	1/9	1/3
邊際機率	1/3	1/3	1/3	1

(a)

$$E(X_1) = E(X_2) = 1\times\frac{1}{3} + 2\times\frac{1}{3} + 3\times\frac{1}{3} = 2$$

(b)

X_1	X_2	$Y=X_1+X_2$	$P(X_1,X_2)$
1	1	2	1/9
1	2	3	1/9
1	3	4	1/9
2	1	3	1/9
2	2	4	1/9
2	3	5	1/9
3	1	4	1/9
3	2	5	1/9
3	3	6	1/9

⇨

Y	P(Y)	Y*P(Y)
2	1/9	2/9
3	2/9	6/9
4	3/9	12/9
5	2/9	10/9
6	1/9	6/9
		4

$$E(Y) = 4$$

(c)

$$E(X_1X_2) = (1\times1 + 1\times2 + 1\times3 + 2\times1 + 2\times2 + 2\times3 + 3\times1 + 3\times2 + 3\times3)\times\frac{1}{9} = 4$$

$$\text{cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 4 - 2\times2 = 0 \quad \Rightarrow \quad X_1, X_2 \text{ 獨立}$$

3.2（95-逢甲-保險）

6. Let X and Y be two continuous random variables with joint density function

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1,\quad 0\le y\le 1 \\ 0 & otherwise \end{cases}$$

Find the conditional probability $P(2X\le 1\mid X+Y\le 1)$.

【解】

$$P(2X\le 1|X+Y\le 1)=\frac{P(2X\le 1\wedge X+Y\le 1)}{P(X+Y\le 1)}$$

$$\begin{aligned}P(X+Y\le 1)&=\int_{x=0}^{1}\int_{y=0}^{1-x}f(x,y)\,dydx=\int_{x=0}^{1}\int_{y=0}^{1-x}(x+y)\,dydx\\&=\int_{x=0}^{1}\left[x(1-x)+\tfrac{1}{2}(1-x)^2\right]dx\\&=\int_{x=0}^{1}\left[-\tfrac{1}{2}x^2+\tfrac{1}{2}\right]dx=-\tfrac{1}{2}\times\tfrac{1}{3}+\tfrac{1}{2}\\&=\tfrac{1}{3}\end{aligned}$$

$$\begin{aligned}P(2X\le 1\wedge X+Y\le 1)&=\int_{x=0}^{\frac{1}{2}}\int_{y=0}^{1-x}f(x,y)\,dydx=\int_{x=0}^{\frac{1}{2}}\int_{y=0}^{1-x}(x+y)\,dydx\\&=\int_{x=0}^{\frac{1}{2}}\left[x(1-x)+\tfrac{1}{2}(1-x)^2\right]dx\\&=\int_{x=0}^{\frac{1}{2}}\left[-\tfrac{1}{2}x^2+\tfrac{1}{2}\right]dx=-\tfrac{1}{2}\times\tfrac{1}{3}\times\tfrac{1}{8}+\tfrac{1}{2}\times\tfrac{1}{4}\\&=\tfrac{5}{48}\end{aligned}$$

$$P(2X\le 1|X+Y\le 1)=\frac{P(2X\le 1\wedge X+Y\le 1)}{P(X+Y\le 1)}=\frac{5/48}{1/3}=\frac{5}{16}$$

3.3（95-淡江-財金）

3.(20%)設有隨機變數 x , y , z , Var(x)=2 , Var(y)=3 , Var(z)=6 , ρ (x , y)= 0.5 , 且 x , y 對 z 爲獨立 ,

(1) 求 Cov (x , y)。

(2) 求 Cov (x , z)。

(3) 求 Var (x + y)。

(4) 求 Var (x + y + 0.8z)。

(5) 求 Cov ($\frac{1}{3}$ (x + y + z) , $\frac{1}{4}$ (x + y))。

【解】

$$\rho(x,y)=\frac{\operatorname{cov}(x,y)}{\sqrt{\operatorname{var}(x)}\sqrt{\operatorname{var}(y)}}$$

$$\operatorname{var}(ax+by)=a^2\operatorname{var}(x)+b^2\operatorname{var}(y)+2ab\operatorname{cov}(x,y)$$

x,y 獨立 $\Leftrightarrow$ $\operatorname{cov}(x,y)=0$ $\Leftrightarrow$ $\rho(x,y)=0$

$$\operatorname{cov}(ax,by)=ab\operatorname{cov}(x,y)$$

$$\operatorname{cov}(x+y,z)=\operatorname{cov}(x,y)+\operatorname{cov}(y,z)$$

(1) $\operatorname{cov}(x,y)=\rho(x,y)\sqrt{\operatorname{var}(x)}\sqrt{\operatorname{var}(y)}=0.5\times\sqrt{2}\times\sqrt{3}=1.2247$

(2) x,z 獨立 $\Rightarrow$ $\operatorname{cov}(x,z)=0$

(3) $\operatorname{var}(x+y)=\operatorname{var}(x)+\operatorname{var}(y)+2\operatorname{cov}(x,y)=2+3+2\times1.2247=7.4495$

(4)

$$\begin{aligned}\operatorname{var}(x+y+0.8z)&=\operatorname{var}(x+y)+\operatorname{var}(0.8z)+\operatorname{cov}(x+y,0.8z)\\&=7.4495+0.8^2\times6+0=11.2895\end{aligned}$$

(5)

$$\operatorname{cov}(x+y+z,x+y)=\operatorname{cov}(x+y,x+y)+\operatorname{cov}(z,x+y)=\operatorname{var}(x+y)=7.4495$$

$$\operatorname{cov}\left(\tfrac{1}{3}(x+y+z),\tfrac{1}{4}(x+y)\right)=\tfrac{1}{3}\times\tfrac{1}{4}\operatorname{cov}(x+y+z,x+y)=\tfrac{1}{12}\operatorname{var}(x+y)=0.6208$$

3.4（95-淡江-保險）

5. Let X and Y be two continuous random variables with joint density function

$$f(x,y)=\begin{cases}4x & 0<x<\sqrt{y}<1\\ 0 & otherwise\end{cases}$$

What is the marginal density function of Y?

(1) $2y$ (2) $2y^2$ (3) y^2 (4) $2\sqrt{y}$ (5) $4y$

【解】

$$f(y)=\int f(x,y)dx=\int_{x=0}^{\sqrt{y}}4xdx=4\times\tfrac{1}{2}x^2\Big|_0^{\sqrt{y}}=2y$$

3.5（95-淡江-保險）

9. The following table gives the joint probability distribution, f(X, Y), of two random variables X and Y.

		X=0	X=1	X=2	X=3
Y	0	0.1	0.2	0	0
	1	0.2	0.25	0.05	0
	2	0	0.05	0.05	0.025
	3	0	0	0.025	0.05

(a) Find the conditional expectation E(X|Y).
(b) Find the variance of X.

【解】

	Y = 0		Y = 1		Y = 2		Y = 3		邊際機率		
x	p(x)	xp(x)	p(x)	xp(x)	p(x)	xp(x)	p(x)	xp(x)	p(x)	xp(x)	$x^2p(x)$
0	0.100	0	0.200	0	0.000	0	0.000	0	0.300	0	0
1	0.200	0.2	0.250	0.25	0.050	0.05	0.000	0	0.500	0.5	0.5
2	0.000	0	0.050	0.1	0.050	0.1	0.025	0.05	0.125	0.25	0.5
3	0.000	0	0.000	0	0.025	0.075	0.050	0.15	0.075	0.225	0.675
	0.300	0.2	0.500	0.35	0.125	0.225	0.075	0.2	1.000	0.975	1.675

(a)

$$E(X|Y)=\begin{cases}\dfrac{0.2}{0.3}=\dfrac{2}{3} & when\ Y=0\\ \dfrac{0.35}{0.5}=\dfrac{7}{10} & when\ Y=1\\ \dfrac{0.225}{0.125}=\dfrac{9}{5} & when\ Y=2\\ \dfrac{0.2}{0.075}=\dfrac{8}{3} & when\ Y=3\end{cases}$$

(b)

$$\operatorname{var}(X)=E(X^2)-[E(X)]^2=1.675-0.975^2=0.7244$$

3.6（95-淡江-保險）

10. The probability density function for the random variable X is given by
$f(x)=mx(1-x)$ for $0\le x\le 1$
$= 0$ elsewhere
(a) Find E(X).
(b) Find the median of X.

【解】

機率密度函數 $f(x)=mx(1-x)$ 在 $x=\frac{1}{2}$ 位置左右對稱，因此

$$E(X)=M_e=\tfrac{1}{2}$$

老老實實積分的話，過程如下：

$$\int f(x)dx=1$$

$$\Rightarrow\quad \int_{x=0}^{1} mx(1-x)dx=m\left(\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right)\Big|_0^1=\tfrac{1}{6}m=1\quad\Rightarrow\quad m=6$$

(a)

$$E(X)=\int xf(x)dx=\int_{x=0}^{1}6x^2(1-x)dx=6\int_{x=0}^{1}\left(x^2-x^3\right)dx=6\left(\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right)\Big|_0^1=\tfrac{1}{2}$$

(b)

$$\int_0^{M_e}f(x)dx=\tfrac{1}{2}\quad\Rightarrow\quad 6\left(\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right)\Big|_0^{M_e}=3M_e^2-2M_e^3=\tfrac{1}{2}\quad\Rightarrow\quad 4M_e^3-6M_e^2+1=0$$

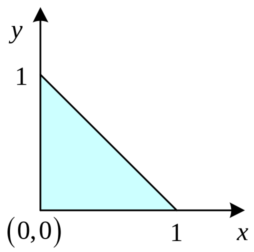
$$\Rightarrow\quad 4M_e^3-6M_e^2+1=\left(2M_e^2+2M_e-1\right)\left(2M_e-1\right)=0\quad\Rightarrow\quad M_e=\tfrac{1}{2}$$

3.7（95-淡江-企管）

1. Suppose that the pair (X,Y) is uniformly distributed over three points (0,0), (0,1) and (1,0).
   (1) Are X and Y independent? **(2%)**
   (2) Find $E(X^2-2\sqrt{Y}+4)$ **(3%)**
   (3) Find $Cov(X,Y)$ **(5%)**

【解】

(x,y)範圍：



令 $f(x,y)=\dfrac{1}{m}$ ，則

$$\int_{x=0}^{1}\int_{y=0}^{1-x}f(x,y)dydx=1\quad\Rightarrow\quad\int_{x=0}^{1}\int_{y=0}^{1-x}\frac{1}{m}dydx=\frac{1}{m}\int_{x=0}^{1}(1-x)dx=\frac{1}{m}\times\frac{1}{2}=1$$

$$\Rightarrow\quad m=\frac{1}{2}$$

(1)

$$f(x)=\int_{y=0}^{1-x}f(x,y)dy=\int_{y=0}^{1-x}2dy=2(1-x)$$

$$f(y)=\int_{x=0}^{1-y}f(x,y)dx=\int_{x=0}^{1-y}2dx=2(1-y)$$

$$f(x,y)=2\neq f(x)f(y)=4(1-x)(1-y)\quad\Rightarrow\quad X,Y\text{ 不獨立}$$

(2)

$$E\left(X^2-2\sqrt{Y}+4\right)=\int_{x=0}^{1}\int_{y=0}^{1-x}2\left(x^2-2\sqrt{y}+4\right)dydx$$
$$=\int_{x=0}^{1}\left[2x^2(1-x)-4\times\frac{2}{3}(1-x)^{\frac{3}{2}}+8(1-x)\right]dx$$
$$=\int_{x=0}^{1}\left[8-8x+2x^2-2x^3-\frac{8}{3}(1-x)^{\frac{3}{2}}\right]dx$$
$$=8-4+\frac{2}{3}-\frac{2}{4}-\frac{8}{3}\times\frac{2}{5}$$
$$=\frac{31}{10}$$

(3)

$$E(X)=\int_{x=0}^{1}2x(1-x)dx=1-\frac{2}{3}=\frac{1}{3}$$

$$E(Y)=\int_{y=0}^{1}2y(1-y)dy=1-\frac{2}{3}=\frac{1}{3}$$

$$E(XY)=\int_{x=0}^{1}\int_{y=0}^{1-x}2xydydx=\int_{x=0}^{1}x(1-x)^2dx=\frac{1}{2}-\frac{2}{3}+\frac{1}{4}=\frac{1}{12}$$

$$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{12}-\frac{1}{3}\times\frac{1}{3}=-\frac{1}{36}$$

3.8（95-淡江-企管）

2. Let X and Y have the joint probability density function

$$f(x,y)=\begin{cases}e^{-x}, & 0\le y\le x;\\ 0, & otherwise\end{cases}$$

(1) Find $E(Y|X=x)$, the conditional mean of Y, given $X=x$. **(5%)**

(2) Find $E(X|Y=y)$, the conditional mean of X, given $Y=y$. **(5%)**

(3) Find ρ, the correlation coefficient of X and Y. **(10%)**

【解】

(1)

$$E(Y|X=x)=\int_{y=0}^{x}ye^{-x}dy=\frac{1}{2}x^2e^{-x}$$

(2)

$$E(X|Y=y)=\int_{x=y}^{\infty}xe^{-x}dx=-xe^{-x}\Big|_{x=y}^{\infty}-\int_{x=y}^{\infty}e^{-x}dx=ye^{-y}+e^{-y}=e^{-y}(1+y)$$
$$\left(f=x,\quad g'=e^{-x}\quad\Rightarrow\quad f'=1,\quad g=-e^{-x}\right)$$

(3)

$$E(X)=\int_{y=0}^{\infty}E(X|Y=y)dy=\int_{y=0}^{\infty}e^{-y}(1+y)dy=\left\{-e^{-y}-e^{-y}(1+y)\right\}\Big|_0^{\infty}=2$$

$$E(Y)=\int_{x=0}^{\infty}E(Y|X=x)dx=\int_{x=0}^{\infty}\frac{1}{2}x^2e^{-x}dx=-\frac{1}{2}x^2e^{-x}+\int_{x=0}^{\infty}xe^{-x}dx=1$$

$$\left(f=\frac{1}{2}x^2,\quad g'=e^{-x}\quad\Rightarrow\quad f'=x,\quad g=-e^{-x}\right)$$

$$E(X^2)=\int_{y=0}^{\infty}\int_{x=y}^{\infty}x^2e^{-x}dxdy=\int_{y=0}^{\infty}(2+2y+y^2)e^{-y}dy=\left\{-e^{-y}(6+4y+y^2)\right\}\Big|_0^{\infty}=6$$

$$E(Y^2)=\int_{y=0}^{\infty}\int_{x=y}^{\infty}y^2e^{-x}dxdy=\int_{y=0}^{\infty}y^2e^{-y}dy=\left\{-e^{-y}(2+2y+y^2)\right\}\Big|_0^{\infty}=2$$

$$E(XY)=\int_{y=0}^{\infty}\int_{x=y}^{\infty}xye^{-x}dxdy=\int_{y=0}^{\infty}y(1+y)e^{-y}dy=\left\{-e^{-y}(3+3y+y^2)\right\}\Big|_0^{\infty}=3$$

$$\sigma_X=\sqrt{E(X^2)-[E(X)]^2}=\sqrt{6-2^2}=\sqrt{2}$$

$$\sigma_Y=\sqrt{E(Y^2)-[E(Y)]^2}=\sqrt{2-1^2}=\sqrt{3}$$

$$\sigma_{XY}=\sqrt{E(XY)-E(X)E(Y)}=\sqrt{3-2\times1}=1$$

$$\rho=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{1}{\sqrt{2}\sqrt{3}}=\frac{1}{\sqrt{6}}$$

3.9（94-雲科大-資管）

3. Consider the following probability density function:

$$f(x)=kx,\qquad 0\le x<2,$$
$$=k(4-x),\qquad 2\le x\le 4,$$
$$=0,\qquad \text{otherwise.}$$

What is the variance of X?

(A) 1/3　(B) 2/3　(C) 1.0　(D) 4/3　(E) 5/3

【解】

先求 k 值

$$\int f(x)dx=1\quad\Rightarrow\quad\int_0^2 kxdx+\int_2^4 k(4-x)dx=4k=1\quad\Rightarrow\quad k=\frac{1}{4}$$

$$E(X)=\int xf(x)dx=\int_0^2\frac{1}{4}x^2dx+\int_2^4\frac{1}{4}x(4-x)dx=\frac{x^3}{12}\Big|_0^2+\left(\frac{x^2}{2}-\frac{x^3}{12}\right)\Big|_2^4=2$$

$$E(X^2)=\int x^2f(x)dx=\int_0^2\frac{1}{4}x^3dx+\int_2^4\frac{1}{4}x^2(4-x)dx=\frac{x^4}{16}\Big|_0^2+\left(\frac{x^3}{3}-\frac{x^4}{16}\right)\Big|_2^4=\frac{14}{3}$$

$$\operatorname{var}(X)=E\left(X^2\right)-\left[E(X)\right]^2=\frac{14}{3}-(2)^2=\frac{2}{3}$$

3.10（94-雲科大-資管）

4. A continuous random variable X has the probability density function

$$f(x)=\frac{2x}{9}, \qquad 0<x<3.$$

What is the value of m such that $P(X \geq m)=P(X \leq m)$?
(A) 1.8383 (B) 2.0 (C) 2.1213 (D) 2.3333 (E) 2.6667

【解】

$$P(X \leq m)=P(X \geq m) \quad \Rightarrow \quad \int_0^m \frac{2}{9}xdx=\int_m^3 \frac{2}{9}xdx \quad \Rightarrow \quad x^2\Big|_0^m=x^2\Big|_m^3 \quad \Rightarrow \quad m=\frac{3\sqrt{2}}{2}$$

3.11（94-雲科大-資管）

6. Let X_1 and X_2 be distributed according to

$$f(x_1,x_2)=2, \qquad 0 \leq x_1 \leq x_2 \leq 1.$$

What is the correlation coefficient between X_1 and X_2?
(A) 0.35 (B) 0.4 (C) 0.45 (D) 0.5 (E) 0.55

【解】

$$E(X_1)=\int_{x_2=0}^{1}\int_{x_1=0}^{x_2} 2x_1 dx_1 dx_2=\frac{1}{3}$$

$$E(X_2)=\int_{x_2=0}^{1}\int_{x_1=0}^{x_2} 2x_2 dx_1 dx_2=\frac{2}{3}$$

$$E\left(X_1^2\right)=\int_{x_2=0}^{1}\int_{x_1=0}^{x_2} 2x_1^2 dx_1 dx_2=\frac{1}{6}$$

$$E\left(X_2^2\right)=\int_{x_2=0}^{1}\int_{x_1=0}^{x_2} 2x_2^2 dx_1 dx_2=\frac{1}{2}$$

$$E(X_1X_2)=\int_{x_2=0}^{1}\int_{x_1=0}^{x_2} 2x_1x_2 dx_1 dx_2=\frac{1}{4}$$

$$\rho=\frac{E(X_1X_2)-E(X_1)E(X_2)}{\sqrt{E\left(X_1^2\right)-\left[E(X_1)\right]^2}\sqrt{E\left(X_2^2\right)-\left[E(X_2)\right]^2}}=\frac{\frac{1}{4}-\frac{1}{3}\times\frac{2}{3}}{\sqrt{\frac{1}{6}-\frac{1}{3}\times\frac{1}{3}}\sqrt{\frac{1}{2}-\frac{2}{3}\times\frac{2}{3}}}=\frac{1}{2}$$

3.12（94-雲科大-資管）

7. For the multivariate distribution

$$f(x,y)=\frac{k}{(1+x+y)^5},\qquad x\geq 0,\ y\geq 0.$$

What is the value of k ?

(A) 12 (B) 14 (C) 16 (D) 18 (E) 20

【解】

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty}k\left(1+x+y\right)^{-5}dxdy=\int_{y=0}^{\infty}\frac{k}{4}\left(1+y\right)^{-4}dy=\frac{k}{12}=1\quad\Rightarrow\quad k=12$$

3.13（94-雲科大-資管）

9. Let X be uniformly distributed between a and b and symmetric about zero with variance 1. What is the value of a^2+b^2 ?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

【解】

$$\begin{cases}\int_a^b kdx=1\\ \int_a^b kxdx=0\\ \int_a^b kx^2dx=1\end{cases}\Rightarrow\begin{cases}k\left(b-a\right)=1\\ \frac{k}{2}\left(b^2-a^2\right)=0\\ \frac{k}{3}\left(b^3-a^3\right)=1\end{cases}\Rightarrow\begin{cases}k=\frac{1}{2\sqrt{3}}\\ a=-\sqrt{3}\\ b=\sqrt{3}\end{cases}\Rightarrow\quad a^2+b^2=6$$

3.14（94-雲科大-資管）

10. A certain type of light bulb has an output known to be normally distributed with mean 2500 end footcandles and standard deviation 75 end footcandles. What is the lower specification limit such that only five percent of the manufactured bulbs will be defective?

(A) 2369.45 fc (B) 2376.63 fc (C) 2383.33 fc

(D) 2389.78 fc (E) 2396.66 fc

【解】

常態分配：$\mu=2500$、$\sigma=75$、左尾、$\alpha=0.05$、求臨界值。

查表 $z=-1.645$，$x^*=\mu+z\sigma=2500-1.645\times 75=2376.63$

3.15（94-雲科大-資管）

11. The Rockwell hardness of a particular alloy is normally distributed with mean 70 and standard deviation 4. Suppose a specimen is acceptable only if its hardness is between 62 and 72. What is the expected number of acceptable specimens among the nine randomly selected specimens?

(A) 5.099 (B) 5.333 (C) 5.667 (D) 6.018 (E) 6.333

【解】

常態分配：$\mu = 70$ 、$\sigma = 4$ 、雙尾、臨界值爲 62、72、求機率。

轉換臨界值：$z_\ell = \dfrac{62-70}{4} = -2$ 、$z_u = \dfrac{72-70}{4} = 0.5$

$P(-2 \le z \le 0.5) = 0.6687$

$n = 9 \times 0.6687 = 6.0183$

3.16（94-雲科大-資管）

12. An assembly consists of three components placed side by side. The length of each component is normally distributed with mean 2 inches and standard deviation 0.2 inches. Specifications require that all assemblies are between 5.7 and 6.3 inches long. On the average, how many assemblies will pass these requirements?

(A) 0.416 (B) 0.456 (C) 0.516 (D) 0.556 (E) 0.616

【解】

常態分配：$\mu = 2 \times 3 = 6$、$\sigma = \sqrt{0.2^2 \times 3} = 0.3464$、雙尾、臨界值爲 5.7、6.3、求機率。

轉換臨界值：$z_\ell = \dfrac{5.7-6}{0.3464} = -0.87$ 、$z_u = \dfrac{6.3-6}{0.3464} = 0.87$

$P(-0.87 \le z \le 0.87) = 0.6157$

3.17（94-雲科大-財金）

2. A certain type of aluminum screen that is 2 feet wide has on the average one flaw in a 100-foot roll. Find the probability that a 50-foot roll has no flaws. ________ (10 points)

【解】

X 爲卜瓦松分配：$\lambda = 0.5$ 、$P(X) = \dfrac{\lambda^x e^{-\lambda}}{x!} = \dfrac{(0.5)^x e^{-0.5}}{x!}$

$P(x=0) = \dfrac{\lambda^x e^{-\lambda}}{x!} = e^{-0.5} = 0.6065$

3.18（94-雲科大-財金）

6. Let the random variables X and Y have the joint p.d.f.

$$f(x,y)=x+y,\quad 0<x<1, 0<y<1.$$
$$=0 \qquad \text{elsewhere.}$$

[a] Compute the variance of X. __________________. (5 points)

[b] Compute the correlation coefficient of X and Y. _____________. (5 points)

【解】

$$E(X)=\int_{y=0}^{1}\int_{x=0}^{1}x(x+y)dxdy=\frac{7}{12},\quad E(Y)=E(X)=\frac{7}{12}$$

$$E(X^2)=\int_{y=0}^{1}\int_{x=0}^{1}x^2(x+y)dxdy=\frac{5}{12},\quad E(Y^2)=E(X^2)=\frac{5}{12}$$

$$E(XY)=\int_{y=0}^{1}\int_{x=0}^{1}xy(x+y)dxdy=\frac{1}{3}$$

(a)

$$\text{var}(X)=E(X^2)-[E(X)]^2=\frac{5}{12}-\frac{7}{12}\times\frac{7}{12}=\frac{11}{144}$$

(b)

$$\text{cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{3}-\frac{7}{12}\times\frac{7}{12}=-\frac{1}{144}$$

$$\rho=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{-1/144}{\sqrt{11/144}\sqrt{11/144}}=-\frac{1}{11}$$

3.19（94-逢甲-經濟）

2. Let X and Y have $E(X)=E(Y)=\mu$ and $Var(X)=Var(Y)=\sigma^2$. Moreover, X and Y are independent. What is the covariance between $X+Y$ and Y? (10%)

【解】

$$\text{cov}(X+Y,Y)=\text{cov}(X,Y)+\text{cov}(Y,Y)=0+\text{var}(Y)=\sigma^2$$

3.20（94-逢甲-保險）

(10分) 某保險公司對旅客的寄送行李的保險理賠，若行李遺失時則賠償$10000元，若毀損時則賠償$1000元。依據此公司對旅客的寄送行李的經驗，每年行李遺失的機率是0.1%，毀損的機率是6%。求每一件行李的純保險費。

【解】

令保費為 x 元

$$\$10,000\times0.1\%+\$1,000\times6\%=x\times(1-0.1\%-6\%)\quad\Rightarrow\quad x=\$74.55$$

3.21（94-逢甲-工工）

1. 隨機變數 X，其機率密度函數(p.d.f.)為

$$f(x)=\begin{cases} x/2, & 0<x\le1 \\ 1/2, & 1<x\le2 \\ (3-x)/2, & 2<x\le3 \\ 0, & 其他 \end{cases}$$

(a) (10 分) 找出累計分配函數(c.d.f.)。
(b) (7 分) 找出 X 的期望值 E(X)。

【解】

(a)

$$F(X)=\int_0^X f(x)dx=\begin{cases} \int_0^X \frac{1}{2}xdx=\frac{1}{4}X^2, & 0<X\le1 \\ \int_0^1\frac{1}{2}xdx+\int_1^X\frac{1}{2}dx=-\frac{1}{4}+\frac{1}{2}X, & 1<X\le2 \\ \int_0^1\frac{1}{2}xdx+\int_1^2\frac{1}{2}dx++\int_2^X\frac{1}{2}(3-x)dx=1-\frac{1}{4}(3-X)^2, & 2<X\le3 \\ \int_0^1\frac{1}{2}xdx+\int_1^2\frac{1}{2}dx++\int_2^3\frac{1}{2}(3-x)dx=1 & 其他 \end{cases}$$

(b)

$$E(X)=\int_0^1 x\frac{x}{2}dx+\int_1^2 x\frac{1}{2}dx+\int_2^3 x\frac{3-x}{2}dx=\frac{1}{6}+\frac{2^2-1}{4}+\frac{3(3^2-2^2)}{4}-\frac{3^3-2^3}{6}=\frac{3}{2}$$

3.22（94-逢甲-工工）

2. 有隨機變數 X，其 p.d.f.為 f(x)。

(a) (5 分) 若要以下函數成為 X 的 p.d.f.，k 值要有如何條件才行。

$$f(x)=(1-k)\,k^x,\ x=0,1,2,\ldots$$

(b) (10 分) 求出在(a)的 f(x)的動差母函數(moment generating function)。

【解】

(a)

$$\sum_{x=0}^{\infty}(1-k)k^x=(1-k)\left[1+k^1+k^2+\cdots\right]=(1-k)\times\frac{1}{(1-k)}=1$$

$1+k^1+k^2+\cdots=\frac{1}{1-k}$ 的條件爲 $k<1$

(b)

$$E\left(e^{tx}\right)=\sum_{x=0}^{\infty}e^{tx}(1-k)k^{x}=(1-k)\left[1+\left(ke^{t}\right)^{1}+\left(ke^{t}\right)^{2}+\cdots\right]=\frac{1-k}{1-ke^{t}}$$

3.23（94-淡江-產經）

3. (十八分；每小題六分) Let the random variable *X* have the p.d.f. *f(x) = x/6, x = 1, 2, 3,* and zero elsewhere.

   a. What is the distribution function for *X*?

   b. What is the mean of *X*?

   c. What is the variance of *X*?

【解】

(a)

$$F(X)=\sum_{-\infty}^{X}f(x)=\begin{cases}0, & X<1\\ 1/6, & 1<X<2\\ 1/2, & 2<X<3\\ 1, & X>3\end{cases}$$

注意，$X=1,2,3$的位置$F(X)$沒有定義。

(b)

$$E(X)=\int xf(x)dx=1\times\tfrac{1}{6}+2\times\tfrac{2}{6}+3\times\tfrac{3}{6}=\tfrac{14}{6}=\tfrac{7}{3}$$

(c)

$$E\left(X^{2}\right)=\int x^{2}f(x)dx=1^{2}\times\tfrac{1}{6}+2^{2}\times\tfrac{2}{6}+3^{2}\times\tfrac{3}{6}=\tfrac{36}{6}=6$$

$$\operatorname{var}(X)=E\left(X^{2}\right)-\left[E(X)\right]^{2}=6^{2}-\left(\tfrac{7}{3}\right)^{2}=\tfrac{275}{9}$$

3.24（94-淡江-國貿）

2) (a) State the probability distribution of a Binomial random variable X with parameters n *and* p. What are the mean and standard deviation of X? (4%)

(b) State the probability distribution of a Normal random variable with mean μ and standard deviation σ. (4%)

(c) When and how a Binomial distribution is well-approximated by a normal distribution ? (8%)

(d) Let X have a Binomial distribution with parameters $n=150$ *and* $p=0.60$. Approximate the probability that $82\le X\le 101$. (6%)

【解】

(a)

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \qquad E(X)=np, \quad \text{var}(X)=np(1-p)$$

(b)

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

(c)

條件： $p\to 0,\ n\to\infty,\ np=\lambda$

$$P(x)=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}\cong\frac{1}{\sqrt{np(1-p)}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-np}{\sqrt{np(1-p)}}\right)^2}$$

(d)

$$\mu=np=150\times 0.6=90, \quad \sigma=\sqrt{np(1-p)}=\sqrt{150\times 0.6\times 0.4}=6$$

$$P(82\le X\le 101)\doteq P(82-0.5\le x\le 101+0.5)$$

$$=P\left(\frac{81.5-90}{6}\le z\le\frac{101.5-90}{6}\right)=P(-1.42\le z\le 1.92)$$

3.25（94-淡江-財金）

1. Assume that you are given the following joint density function,

$$f(x,y)=\begin{cases}8xy, & 0\le x\le 1,\ 0\le y\le x\\ 0, & \text{otherwise}\end{cases}$$

A. (10%) Please find $E(X)$.

B. (10%) Please find $V(X)$.

【解】

(a)

$$E(X)=\int_{x=0}^{1}\int_{y=0}^{x}x(8xy)\,dydx=\int_{x=0}^{1}4x^2x^2dx=\frac{4}{5}$$

(b)

$$E(X^2)=\int_{x=0}^{1}\int_{y=0}^{x}x^2(8xy)\,dydx=\int_{x=0}^{1}4x^3x^2dx=\frac{4}{6}=\frac{2}{3}$$

$$\text{var}(X)=E(X^2)-[E(X)]^2=\frac{2}{3}-\left(\frac{4}{5}\right)^2=\frac{2}{75}$$

3.26（94-淡江-保險）

一、獨立地投擲一公正的銅板直到出現正面為止，觀測其出現正反面的情形。
(1) 試寫出其樣本空間（sample space）。（5分）
(2) 若令X表示所需投擲的總次數，
試求（a）X之機率密度函數（pdf）。（5分）
（b）X之期望值E（X）。（5分）
（c）P(X = 1,3,5, ・・・)。（5分）

【解】

(1)

$$S=\{+,-+,--+,---+,\ldots\}$$

(2)

$$f(x)=\left(\frac{1}{2}\right)^x$$

$$E(X)=\sum_{x=1}^{\infty}x\left(\frac{1}{2}\right)^x=2$$

$$P(X=1,3,5,\ldots)=\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\left(\frac{1}{4}\right)^2+\cdots=\frac{1/2}{1-1/4}=\frac{2}{3}$$

3.27（94-政大-財管）

2. (15 points)
(A). What is the probability that none of the 140 students are born on October 10th ? Assume all 365 days are equally likely.
(B). A six-sided die is tossed repeatedly. How many rolls on the average does it take for all six sides to have appeared?
(C). Suppose you have a sample data set consisting of 10 observations. If the value of the smallest observation is decreased by 1 unit, how of the following will change: mean, median, mode, and why?

【解】

(A)機率分配的部分，不要用超幾何分配，用二項分配作，而且以卜瓦松來近似。

$$\lambda=\frac{140}{365}\text{ ， }P(x=0)=\frac{\lambda^x e^{-\lambda}}{x!}=e^{-\frac{140}{365}}=0.6813$$

(B)六個階段的幾何分配，其成功機率 p 分別為 $\frac{6}{6},\frac{5}{6},\ldots,\frac{1}{6}$ ，全部次數加總即答案。

$$E(x)=\frac{1}{6/6}+\frac{1}{5/6}+\frac{1}{4/6}+\frac{1}{3/6}+\frac{1}{2/6}+\frac{1}{1/6}=14.7$$

(C)平均數減少$\frac{1}{10}$，中位數不變，眾數的變化較複雜。

3.28（94-政大-財管）

3. (10 points)

(A). Suppose that Y_1 is $Binomial(n, p_1)$. Show that

$$\frac{(Y_1-np_1)^2}{np_1(1-p_1)}=\frac{(Y_1-np_1)^2}{np_1}+\frac{(Y_2-np_2)^2}{np_2}$$

Where $Y_2=n-Y_1$ and $p_2=1-p_1$

(B). The random variables X and Y have joint density

$$f_{X,Y}(x,y)=3xe^{-x(3+y)}$$

a. The marginal distribution of X is?

b. The conditional density of Y given X is?

c. The conditional probabilities $Pr(Y<2 \mid X=3)$ is?

【解】

(a)

$$\frac{1}{np_1}+\frac{1}{np_2}=\frac{1}{np_1}+\frac{1}{n(1-p_1)}=\frac{np_1+n(1-p_1)}{n^2p_1(1-p_1)}=\frac{1}{np_1(1-p_1)}$$

$$(Y_2-np_2)^2=\left(n-Y_1-n(1-p_1)\right)^2=(-Y_1+np_1)^2=(Y_1-np_1)^2$$

$$\Rightarrow\quad \frac{(Y_1-np_1)^2}{np_1}+\frac{(Y_2-np_2)^2}{np_2}=\frac{(Y_1-np_1)^2}{np_1}+\frac{(Y_1-np_1)^2}{np_2}=\frac{(Y_1-np_1)^2}{np_1(1-p_1)}$$

(b)

本題需在$x>0$ 、 $y>0$的範圍限制下， $f_{X,Y}(x,y)$才是聯合機率密度函數。

$$f_X(x)=\int_0^\infty 3xe^{-x(3+y)}dy=\frac{e^{-3x}}{x}$$

$$f(y|X=x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\frac{3xe^{-x(3+y)}}{e^{-3x}/x}=3x^2e^{-xy}$$

$$P(Y<a|X=x)=\int_0^a f(y|X=x)dy=\int_0^a 3x^2e^{-xy}dy=3x\left(1-e^{-ax}\right)$$

$$\Rightarrow\quad P(Y<2|X=3)=3\times3\left(1-e^{-2\times3}\right)=9\left(1-e^{-6}\right)$$

3.29（94-政大-風管）

> 6. If Y has an exponential distribution, show that
>
> $$P(Y \ge t+T \mid Y \ge T) = P(Y \ge t)\text{. } \textbf{(10 points)}$$

【解】

指數分配的機率密度函數如下：

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0, \lambda > 0$$

則

$$P(Y \ge T) = \int_T^{\infty} \lambda e^{-\lambda y} dy = e^{-\lambda T}, \qquad P(Y \ge t) = e^{-\lambda t}$$

$$P(Y \ge t+T \text{ 且 } Y \ge T) = P(Y \ge t+T) = e^{-\lambda(t+T)}$$

因此

$$P(Y \ge t+T \mid Y \ge T) = \frac{P(Y \ge t+T \text{ 且 } Y \ge T)}{P(Y \ge T)} = \frac{e^{-\lambda(t+T)}}{e^{-\lambda T}} = e^{-\lambda t} = P(Y \ge t)$$

3.30（94-政大-風管）

> 7. The $p.d.f$ of the random variable X is given by
>
> $$f(x) = \begin{cases} \dfrac{k}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$
>
> Find
>
> (a) the value of k; **(4 points)**
>
> (b) $P(X < \frac{1}{4})$ and $P(X > 1)$. **(6 points)**

【解】

(a)

$$\int_0^4 \frac{k}{\sqrt{x}} dx = 4k = 1 \quad \Rightarrow \quad k = \frac{1}{4}$$

$$\left(\int_0^4 \frac{k}{\sqrt{x}} dx = k\int_0^4 x^{-\frac{1}{2}} dx = k \frac{1}{\frac{1}{2}} x^{\frac{1}{2}} \Big|_0^4 = 4k \right)$$

(b)

$$P\left(X<\tfrac{1}{4}\right)=\int_0^{\frac{1}{4}}\frac{1}{4\sqrt{x}}dx=\tfrac{1}{2}x^{\frac{1}{2}}\Big|_0^{\frac{1}{4}}=\tfrac{1}{4}$$

$$P\left(X>1\right)=\int_1^{4}\frac{1}{4\sqrt{x}}dx=\tfrac{1}{2}x^{\frac{1}{2}}\Big|_1^{4}=\tfrac{1}{2}$$

3.31（94-政大-風管）

> 8. A random variable X has its probability density given by
>
> $$f(x)=\begin{cases}2\beta xe^{-\beta x^2} & \text{for } x>0\\ 0 & \text{otherwise}\end{cases}$$
>
> where $\beta>0$. Show that for this distribution
>
> (a) $\mu=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}$; **(7 points)**
>
> (b) $\sigma^2=\frac{1}{\beta}(1-\frac{\pi}{4})$. **(8 points)**

【解】

(a)

$$\mu=E\left(X\right)=\int_0^{\infty}x2\beta xe^{-\beta x^2}dx=-xe^{-\beta x^2}\Big|_0^{\infty}+\int_0^{\infty}e^{-\beta x^2}dx=\int_0^{\infty}e^{-\beta x^2}dx$$

其中，$f=x,\ g'=2\beta xe^{-\beta x^2}\quad\Rightarrow\quad f'=1,\ g=-e^{-\beta x^2}$

令 $I=\int_0^{\infty}e^{-\beta x^2}dx=\int_0^{\infty}e^{-\beta y^2}dy$ ，則 $I^2=\int_0^{\infty}\int_0^{\infty}e^{-\beta x^2}e^{-\beta y^2}dxdy=\int_0^{\infty}\int_0^{\infty}e^{-\beta\left(x^2+y^2\right)}dxdy$

令 $x=r\sin\theta,\ y=r\cos\theta$ 則 $J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r}\\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\sin\theta & \cos\theta\\ r\cos\theta & -r\sin\theta\end{vmatrix}=-r,\ dxdy=\left|J\right|drd\theta=rdrd\theta$

$$I^2=\int_0^{\infty}\int_0^{\infty}e^{-\beta\left(x^2+y^2\right)}dxdy=\int_0^{\infty}\int_0^{\frac{1}{2}\pi}e^{-\beta r^2}rd\theta dr=\int_0^{\infty}re^{-\beta r^2}\left(\int_0^{\frac{1}{2}\pi}d\theta\right)dr=\tfrac{1}{2}\pi\int_0^{\infty}re^{-\beta r^2}dr=\frac{\pi}{4\beta}$$

$$\Rightarrow\quad I=\sqrt{\frac{\pi}{4\beta}}\quad\Rightarrow\quad\mu=\sqrt{\frac{\pi}{4\beta}}=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}$$

(b)

$$E\left(X^2\right)=\int_0^{\infty}x^2 2\beta xe^{-\beta x^2}dx=\int_0^{\infty}\frac{1}{\beta}ue^{-u}du=-\frac{1}{\beta}e^{-u}\Big|_0^{\infty}=\frac{1}{\beta}$$

其中，$u=\beta x^2,\ du=2\beta xdx$

$$\sigma^2=E\left(X^2\right)-\left[E\left(X\right)\right]^2=\frac{1}{\beta}-\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\right)^2=\frac{1}{\beta}-\frac{\pi}{4\beta}=\frac{1}{\beta}\left(1-\frac{\pi}{4}\right)$$

3.32（94-政大-企管）

3. A restaurant manager wants to know the pattern of customers' arrival. Based on past experience, it is assumed that the number of customers per minute follows a Poisson distribution with an average of 0.5 per minute. What is the probability that in
   a. one minute there will be at least 2 customers? (6 %)
   b. three minutes there will be less than or equal to 2 customers? (6 %)
   c. five minutes there will be exactly 3 customers? (6 %)

【解】

X 爲卜瓦松分配：$\lambda = 0.5$（人/1 分鐘），$P(X) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

(a)

卜瓦松分配：$\lambda = 0.5$（人/1 分鐘）

$$P(x \geq 2) = 1 - P(x=0) - P(x=1) = 1 - \frac{0.5^0 e^{-0.5\times 0}}{0!} - \frac{0.5^1 e^{-0.5}}{1!} = 0.0902$$

(b)

卜瓦松分配：$\lambda = 0.5 \times 3 = 1.5$（人/3 分鐘）

$$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2) = e^{-1.5} + 1.5e^{-1.5} + \frac{1.5^2 e^{-1.5}}{2} = 0.8088$$

(c)

卜瓦松分配：$\lambda = 0.5 \times 5 = 2.5$（人/5 分鐘）

$$P(x=3) = \frac{2.5^3 e^{-2.5}}{3!} = 0.2138$$

3.33（94-台大-商學）

1. The starting salaries of individuals with an MBA degree are normally distributed with a mean of \$40,000 and a standard deviation of \$5,000. What is the probability that a randomly selected individual with an MBA degree will get a starting salary of at least \$30,000?

   A. 0.4772
   B. 0.9772
   C. 0.0228
   D. 0.5000

【解】

X 爲常態分配：$\mu = 40,000$ 、$\sigma = 5,000$

$$z \geq \frac{30,000 - 40,000}{5,000} = -2$$

$P(X \geq 30{,}000) = P\left(z \geq \dfrac{30{,}000 - 40{,}000}{5{,}000} = -2\right) = 0.9772$ （查表）

也可從 $P(-2 \leq z \leq 2) = 0.95$ 來計算。

3.34（94-台大-商學）

9. Which of the following is a characteristic of a binomial experiment?

   A. At least 2 outcomes are possible
   B. The probability of success changes from trial to trial
   C. The trials are independent
   D. All of these answers are correct.

【解】

X 爲二項分配：n 、 p 、 $P(X) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$， $x = 0, 1, 2, \ldots, n$

(A)真，$n \geq 1 \Rightarrow$ 至少有 $x = 0, x = 1$

(B)僞，p 需爲常數

(C)僞，實驗間需獨立

(D)僞

3.35（94-台大-商學）

10. The function that defines the probability distribution of any continuous random variable is

   A. Normal function
   B. Uniform function
   C. Both the normal function and the uniform function are correct.
   D. Probability density function

【解】

(D)

3.36（94-台大-商學）

13. If the correlation coefficient between X and Y is equal to 1, then

   A. Covariance between X and Y is greater than 0
   B. Covariance between X and Y is equal to 1
   C. Covariance between X and Y can be any real number
   D. Y = f(X)

【解】

$$\rho = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = 1 \quad \Rightarrow \quad \text{cov}(X,Y) = \sigma_X \sigma_Y > 0$$ ， 選(A)

3.37（94-台大-商學）

14. A campus program evenly enrolls undergraduate and graduate students. If a random sample of 4 students is selected from the program to be interviewed about the introduction of a new fast food outlet on the ground floor of the campus building, what is the probability that all 4 students selected are undergraduate students?

A. 0.0256
B. 0.0625
C. 0.16
D. 1.00

【解】

X 為二項分配： $n = 4$ 、 $p = 0.5$ 、 $P(X) = C_x^n p^x (1-p)^{n-x} = \frac{4!}{x!(4-x)!}(0.5)^4$

$P(x=4) = 0.5^4 = 0.0625$

3.38（93-雲科大-資管）

5. Given that $E(X+4) = 10$ and $E[(X+4)^2] = 116$. What is the standard deviation of X?

(A) 4 (B) 8 (C) 12 (D) 16 (E) 20

【解】

$$\begin{cases} E(X+4) = E(X) + 4 = 10 \\ E\left[(X+4)^2\right] = E(X^2) + 8E(X) + 16 = 116 \end{cases} \Rightarrow \begin{cases} E(X) = 6 \\ E(X^2) = 52 \end{cases}$$

$$\sigma_X = \sqrt{E(X^2) - [E(X)]^2} = \sqrt{52 - 6^2} = 4$$

或

$$\sigma_X = \sqrt{E\left[(X+4)^2\right] - [E(X+4)]^2} = \sqrt{116 - 10^2} = 4$$

（平移不會影響隨機變數的變異數、標準差。）

3.39（93-雲科大-資管）

6. The probability density function of a random variable Y is $f(y) = k / y^2$, for $1 < y < \infty$. What is the value of k?

(A) 0.5 (B) 1.0 (C) 1.25 (D) 1.5 (E) 2.0

【解】

$$\int_1^{\infty}\frac{k}{y^3}dy=\frac{k}{-2}y^{-2}\Big|_1^{\infty}=\frac{k}{2}\quad\Rightarrow\quad k=2$$

3.40（93-雲科大-資管）

> 7. A random variable X has a binomial distribution with mean 6 and variance 3.6. What is the probability that $X=4$?
>
> (A) 0.1059 (B) 0.1268 (C) 0.1493 (D) 0.1625 (E) 0.1842

【解】

X 為二項分配：$np=6$、$np(1-p)=3.6 \quad\Rightarrow\quad p=0.4, n=15$

$$P(X=4)=\frac{15!}{4!11!}0.4^4 0.6^{11}=0.1268$$

3.41（93-雲科大-資管）

> 8. One of four different prizes was randomly put into each box of a cereal. If a family decided to buy this cereal until it contained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?
>
> (A) 6.33 (B) 7.33 (C) 8.33 (D) 9.33 (E) 10.33

【解】

令$T=X_1+X_2+X_3+X_4$，其中X_1為出現第一種獎品的次數，$X_i, i=2,3,4$為出現第$i-1$種獎品後到出現第i種獎品出現的次數。則X_1、X_2、X_3、X_4分別為成功機率$\frac{4}{4}$、$\frac{3}{4}$、$\frac{2}{4}$、$\frac{1}{4}$的幾何分配。因此

$$E(X_1)=\frac{1}{p_1}=1,\quad E(X_2)=\frac{1}{p_2}=\frac{4}{3},\quad E(X_3)=\frac{1}{p_3}=\frac{4}{2},\quad E(X_4)=\frac{1}{p_4}=\frac{4}{1}$$

故

$$E(T)=E(X_1)+E(X_2)+E(X_3)+E(X_4)=\frac{4}{4}+\frac{4}{3}+\frac{4}{2}+\frac{4}{1}=8.33$$

3.42（93-雲科大-資管）

9. Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If the Poisson distribution is assumed, what is the probability of at most one flaw in 225 square feet?

(A) 0.375 (B) 0.417 (C) 0.462 (D) 0.509 (E) 0.558

【解】

X 爲卜瓦松分配：$\lambda = \frac{225}{150} = 1.5$ 、$P(X) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(1.5)^x e^{-1.5}}{x!}, \quad x = 0,1,2,\ldots$

$P(X \le 1) = P(X = 0) + P(X = 1) = e^{-1.5} + 1.5e^{-1.5} = 0.5578$

3.43（93-雲科大-資管）

10. Let the random variable X have an exponential distribution with density function $f(x) = \lambda e^{-\lambda x}$, for $x > 0$, and the random variable $Y = 1 - e^{-\lambda X}$. What is the variance of Y?

(A) 0.0833 (B) 0.1056 (C) 0.1424 (D) 0.1839 (E) 0.2061

【解】

$E(Y) = \int_0^\infty (1 - e^{-\lambda x}) \lambda e^{-\lambda x} dx = \int_0^1 u\,du = \frac{1}{2}$

其中，$u = 1 - e^{-\lambda x}, \ du = \lambda e^{-\lambda x}, \ u \in (0,1)$

$E(Y^2) = \int_0^\infty (1 - e^{-\lambda x})^2 \lambda e^{-\lambda x} dx = \int_0^1 u^2 du = \frac{1}{3}$

$\text{var}(Y) = E(X^2) - [E(X)]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} = 0.0833$

3.44（93-雲科大-資管）

11. If the moment-generating function of X is $M_X(t) = (1-t)^{-2}$, for $t < 1$, what is the variance of X?

(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

【解】

$m(t) = (1-t)^{-2}, \qquad m'(t) = 2(1-t)^{-3}, \quad m''(t) = 6(1-t)^{-4}$

$E(X) = m'(t=0) = 2, \quad E(X^2) = m''(t=0) = 6$

$\text{var}(X) = E(X^2) - [E(X)]^2 = 6 - 2^2 = 2$

3.45（93-雲科大-財金）

1. Given the joint p.d.f. $f(x,y)=6(1-x-y)$ for $0<y<1-x$ and $x>0$.
Find $f(y|x)$. ______________. (10 points)

【解】

$$f(x)=\int_0^{1-x} f(x,y)\,dy=\int_0^{1-x}6(1-x-y)\,dy=3(1-x)^2$$

$$f(y|x)=\frac{f(x,y)}{f(x)}=\frac{6(1-x-y)}{3(1-x)^2}=\frac{2(1-x-y)}{(1-x)^2}$$

3.46（93-雲科大-財金）

2. Find the variance for the continuous variable Y with p.d.f.

(a) $f(y)=\frac{1}{2}e^{-|y|}$. __________. (5 points)

(b) $f(y)=6y(1-y)$, $0<y<1$. __________. (5 points)

【解】

(a)

$$E(Y)=\int_{-\infty}^{0} y\tfrac{1}{2}e^{y}dy+\int_0^{\infty} y\tfrac{1}{2}e^{-y}dy=\left(-\frac{1}{2}(1-y)e^{y}\right)\Bigg|_{-\infty}^{0}+\left(\frac{1}{2}(1+y)e^{-y}\right)\Bigg|_{0}^{\infty}=-\frac{1}{2}+\frac{1}{2}=0$$

$$E(Y^2)=\int_{-\infty}^{0} y^2\tfrac{1}{2}e^{y}dy+\int_0^{\infty} y^2\tfrac{1}{2}e^{-y}dy$$

$$=\left(\frac{1}{2}(2-2y+y^2)e^{y}\right)\Bigg|_{-\infty}^{0}-\left(\frac{1}{2}(2+2y+y^2)e^{-y}\right)\Bigg|_{0}^{\infty}=1+1=2$$

$$\text{var}(Y)=E(Y^2)-[E(Y)]^2=2$$

(b)

$$E(Y)=\int_0^1 y6y(1-y)dy=\left(2y^3-\frac{3}{2}y^4\right)\Bigg|_0^1=\frac{1}{2}$$

$$E(Y^2)=\int_0^1 y^2 6y(1-y)dy=\left(\frac{3}{2}y^4-\frac{6}{5}y^5\right)\Bigg|_0^1=\frac{3}{10}$$

$$\text{var}(Y)=E(Y^2)-[E(Y)]^2=\frac{3}{10}-\left(\frac{1}{2}\right)^2=\frac{1}{20}$$

3.47（93-雲科大-財金）

3. Evaluate the following expressions:

$$\sum_{k=0}^{10}k\binom{10}{10-k}(0.5)^{10}.$$ __________________. (10 points)

【解】

X 爲二項分配：$n=10$ 、 $p=0.5$ 、 $P(X)=C_x^n p^x (1-p)^{n-x} = \dfrac{10!}{x!(10-x)!}(0.5)^n$

$$E(X)=\sum_{k=0}^{10} k\binom{10}{10-k}(0.5)^{10} = np = 10\times 0.5 = 5$$

或

$$\sum_{k=0}^{10} k\binom{10}{10-k}(0.5)^{10} = \sum_{k=0}^{10} k\frac{10!}{k!(10-k)!}(0.5)^{10} = k\frac{10}{k}\times 0.5\sum_{k=1}^{10}\frac{9!}{(k-1)![9-(k-1)]!}(0.5)^9$$

$$=10\times 0.5\sum_{j=0}^{9}\frac{9!}{j![9-j]!}(0.5)^9 = 10\times 0.5 = 5$$

3.48（93-雲科大-財金）

> 5. The net profit of an investment is normally distributed with a mean of $10,000 and a standard deviation of $5,000. The probability that the investor's net gain will be at least $5,000 is ________. (10 points)

【解】

X 爲常態分配：$\mu=10{,}000$ 、 $\sigma=5{,}000$

$$P(X\ge 5{,}000)=P\left(z\ge\frac{5{,}000-10{,}000}{5{,}000}=-2\right)=1-\frac{5\%}{2}=0.975$$

（應用 $P(-2\le z\le 2)=95\%$ 之經驗公式。）

3.49（93-雲科大-財金）

> 6. If X is a random variable with mean 100 and standard deviation 35, find a number c such that the probability that the random variable X deviates from mean in the c is at least 0.86. ________________. (10 points)

【解】

X 爲常態分配：$\mu=100$ 、 $\sigma=35$

$P(500-c\le X\le 500+c)=0.86$

$\Rightarrow\ P\left(\dfrac{-c}{35}\le\dfrac{X-500}{35}=z\le\dfrac{c}{35}\right)=0.86=P(-1.4758\le z\le 1.4758)$ （查表）

$\Rightarrow\ \dfrac{c}{35}=1.4758\ \Rightarrow\ c=35\times 1.4758=51.65$

3.50（93-雲科大-企管）

3. 某公司估計其顧客到達櫃台的機率，發現 1 秒鐘內有 1 位顧客到達櫃台的機率爲 0.1，假設顧客到櫃台是隨機的(即顧客到櫃台都是相互獨立的)，請問：
(1)第 1 位顧客在第 3 秒到達的機率？__________(5%)
(2)第 1 位顧客在第 2 秒之後才到達的機率？__________(5%)

【解】

T 爲指數分配：$\lambda = 0.1$（人/秒）、 $f(T) = \lambda e^{-\lambda t},\quad t > 0$

(1)

$$P(2 < T < 3) = \int_2^3 0.1e^{-0.1t}dt = -e^{-0.1t}\Big|_2^3 = 0.0779$$

(2)

$$P(T > 2) = \int_2^\infty 0.1e^{-0.1t}dt = -e^{-0.1t}\Big|_2^\infty = 0.8187$$

3.51（93-逢甲-經濟）

一.(5 分) 每日股票價格可能上漲、下跌或平盤，三種狀況機率均等。請問一支股票在 6 個交易日裡正好上漲 4 次的機率為多少？

【解】

X 爲二項分配：$n = 6$ 、 $p = \dfrac{1}{3}$ 、$P(X) = C_x^n p^x (1-p)^{n-x} = \dfrac{6!}{x!(6-x)!}\left(\dfrac{1}{3}\right)^x \left(\dfrac{2}{3}\right)^{6-x}$

$$P(x=4) = C_4^6 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 = 15 \times \frac{4}{729} = \frac{60}{729}$$

3.52（93-逢甲-經濟）

三. (10 分) 假如歐式賣權(P)的價格公式如下：

$P = S[\Phi(x_1)-1] - K e^{-RT}[\Phi(x_2)-1]$,

其中 S =股價,
K =履約價,
R =利率,
T =到期期間,
$x_2 = x_1 - \sigma T^{1/2}$,
σ =股價的標準差,
e =2. 71828,
$\Phi(x_1), \Phi(x_2)$ =累計標準常態分配函數.

假設 S=47, K=50, T=0.50, R=0.10, σ=0.40, x_1=0.10，請算出此歐式賣權(P)的價格?

【解】

$x_2 = x_1 - \sigma T^{1/2} = 0.1 - 0.4 \times \sqrt{0.5} = -0.1828$

查表 $\Phi(x_1) = \Phi(0.1) = 0.5398, \quad \Phi(x_2) = \Phi(-0.1828) = 0.4275$

$P = S[\Phi(x_1) - 1] - Ke^{-RT}[\Phi(x_2) - 1] = 47[0.5398 - 1] - 50e^{-0.1 \times 0.5}[0.4275 - 1] = 5.60$

3.53（93-逢甲-工工）

1. 以下每題 6 分，共計 30 分。詳細敘述每題解題過程，否則不給分。
   (1) 已知事件 A 與 B 獨立、P(A∪B) = 0.6、P(A) = 0.3，求 P(B)。
   (2) 指數分配隨機變數具有無記憶性質，用一個實際例子說明之。
   (3) 自[0, 1]任取一數 u，說明如何用 u 產生期望值為 3 的指數分配隨機變數值。
   (4) 自[0, 2]任取兩個數，求兩個數的和大於 1 的機率。
   (5) 自 N(0, 1)取樣本大小 11 的隨機樣本，求樣本中位數為負的機率。

【解】

(1) 獨立 $\Rightarrow P(A \cap B) = P(A)P(B)$

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) \quad \Rightarrow \quad 0.3 + P(B) = 0.6 + 0.3P(B) \quad \Rightarrow \quad P(B) = \frac{3}{7}$$

(2) 第三章內容。

等車時間為指數分配，$\frac{1}{\lambda} = 10$ 分鐘，若已經等 8 分鐘，則平均需再等還是 10 分鐘。

(3) 屬 OR 系統模擬內容。

(4) 第三章內容。

$$f(x,y) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(x + y \le 1) = \int_{x=0}^{1} \int_{y=0}^{1-x} f(x,y)\,dydx = \frac{1}{4} \int_{x=0}^{1} (1-x)\,dx = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$$P(x + y > 1) = 1 - P(x + y \le 1) = 1 - \frac{1}{8} = \frac{7}{8}$$

(5) 第三章內容。

標準常態為平均零之對稱分配，因此中位數為正、為負的機率相等。

計算過程如下：

二項分配：$n = 11$、 $p = 0.5$ 、 x ：小於零次數

$$P(x \ge 6) = \left(C_6^{11} + C_7^{11} + C_8^{11} + C_9^{11} + C_{10}^{11} + C_{11}^{11}\right) \times 0.5^{11} = \frac{1024}{2048} = \frac{1}{2}$$

3.54（93-逢甲-工工）

2. 第一條與第二條生產線的不良率分別為 0.03 與 0.02。兩條生產線各任取 3 件產品檢驗，求總共檢驗 6 件產品中有 3 件不良品的機率。(8 分)

【解】

令 X_1 、 X_2 分別爲來自各生產線的不良品數量，則

X_1 爲二項分配： $n_1 = 3$ 、 $p_1 = 0.03$

X_2 爲二項分配： $n_2 = 3$ 、 $p_2 = 0.02$

$$\begin{aligned}P(X_1+X_2=3) &= P(X_1=0)P(X_2=3)+P(X_1=1)P(X_2=2)\\ &\quad +P(X_1=2)P(X_2=1)+P(X_1=3)P(X_2=0)\\ &=\left(0.03^0\times0.97^3\right)\times\left(0.02^3\times0.98^0\right)+\left(3\times0.03^1\times0.97^2\right)\times\left(3\times0.02^2\times0.98^1\right)\\ &\quad+\left(3\times0.03^2\times0.97^1\right)\times\left(3\times0.02^1\times0.98^2\right)+\left(0.03^3\times0.97^0\right)\times\left(0.02^0\times0.98^3\right)\\ &=\end{aligned}$$

3.55（93-逢甲-工工）

3. 自區域{(x, y) | 0 ≤ x ≤ 10, 10 - x ≤ y ≤ 14 - x}內任取一點的座標為(X, Y)。
(1) 求(X, Y)的 j.p.d.f.。(5 分)
(2) 求 Y 的 m.p.d.f.。(7 分)

【解】

(1)

均等分配，令 $f(x,y)=m,\quad 0\le x\le 10, -x\le y\le 14-x$

$$\int_0^{10}\int_{-x}^{14-x} m\,dy\,dx = 1 \quad\Rightarrow\quad m\int_0^{10}(14-x+x)\,dx = 140m = 1 \quad\Rightarrow\quad m=\frac{1}{140}$$

聯合機率密度函數 $f(x,y)=\dfrac{1}{140},\quad 0\le x\le 10, -x\le y\le 14-x$

(2)

$$f(y)=\int_0^{10} f(x,y)\,dx=\int_0^{10}\frac{1}{140}\,dx=\frac{1}{14}$$

3.56（93-淡江-國貿）

三、(15%)已知某系有 1000 位學生，其中有 100 位女生。今欲隨機抽出 20 位學生參加社區服務；令 X 表示抽出之 20 位學生中女生的人數。
(1) 求 X 之機率分配。
(2) 求抽出 20 位學生中恰有 6 位女生的機率？(只須列出求機率之數學式)
(3) X 之近似機率分配爲何？

【解】

(1)

X 爲超幾何分配： $N=1{,}000$ 、 $S=200$ 、 $n=20$ 、 $P(X)=\dfrac{C_x^S C_{n-x}^{N-X}}{C_n^N},\quad x=0,1,2,\ldots,n$

(2)

$$P(X=6)=\frac{C_6^{200}C_{14}^{800}}{C_{20}^{1{,}000}}$$

(3)

以二項分配近似： $n=20$ 、 $p=\dfrac{S}{N}=0.2$ 、 $P(X)=C_x^n p^x(1-p)^{n-x},\quad x=0,1,2,\ldots,n$

$$P(X=6)=\frac{20!}{6!14!}(0.2)^6(0.8)^{14}=0.1091$$

3.57（93-淡江-財金）

(1) Suppose a certain mutual fund has an annual rate of return that is approximately normally distributed with mean (expected value) 5% and standard deviation 4%. What is the probability that your 1-year return will exceed 10%. (From Table of the Standard Normal Distribution, area under Z=2.5 is 0.4938 and Z=1.25 is 0.3944)
(a) 0.0062 (b) 0.1056 (c) 0.5062 (d) 0.6056

【解】

X 爲常態分配： $\mu=5\%$ 、 $\sigma=4\%$

$$P(x\geq 10\%)=P\left(z\geq\frac{10\%-5\%}{4\%}=1.25\right)=0.5+0.3944=0.8944$$

3.58（93-淡江-財金）

(2) Suppose that X is a random variable for which E(X)=1, E(X^2)=4, and E(X^3)=10. Find the value of the third central moment of X ?

【解】

$$\begin{aligned}E\left[(X-\mu)^3\right]&=E\left(X^3-3\mu X^2+3\mu^2X-\mu^3\right)\\&=E\left(X^3\right)-3\mu E\left(X^2\right)+3\mu^2E(X)-\mu^3\\&=10^3-3\times1\times4+3\times1^2\times1-1^3\\&=990\end{aligned}$$

3.59（93-淡江-保險）

3. What outcomes are in the event "A fair coin is to be tossed until a head comes up for the first time"? Find the probability of that happening on an even-numbered toss.

【解】

$H, TH, TTH, TTTH, \dots$

幾何分配： $p = 0.5$ 、 $P(X) = p(1-p)^{n-x} = (0.5)^x$

$P(X\text{為偶數}) = P(2) + P(4) + \cdots = (0.5)^2 + (0.5)^4 + \cdots = \frac{0.25}{1-0.25} = \frac{1}{3}$

3.60（93-淡江-保險）

6. Consider an experiment consisting of flipping a fair coin three times. Let X denote the number of heads on the first flip; Y, the total number of heads for the three tosses.
(a) Find $F_Y(y)$.
(b) Find E(Y|X).
(c) Find the covariance of X and Y.

【解】

X 為白努力分配： $p = 0.5$ 、 $P(X) = p^x(1-p)^{1-x} = (0.5)^1, \quad x = 0,1$

Y 為二項分配：$n = 3$ 、 $p = 0.5$ 、$P(Y) = C_y^n p^y (1-p)^{n-y} = \frac{3!}{y!(3-y)!}(0.5)^3, \quad y = 0,1,2,3$

(a)

$P(Y=0) = 0.125, \quad P(Y=1) = 0.375, \quad P(Y=2) = 0.375, \quad P(Y=3) = 0.125$

$$F(Y) = P(Y \geq y) = \begin{cases} 0.125, & 0 \leq y < 1 \\ 0.5, & 1 \leq y < 2 \\ 0.875, & 2 \leq y < 3 \\ 1, & 3 \leq y \end{cases}$$

(b)

$$P(Y|X=0) = \begin{cases} C_y^2 (0.5)^2 = \frac{2}{y!(2-y)!}(0.5)^2, & y = 0,1,2 \\ 0, & otherwise \end{cases}$$

$$P(Y|X=1) = \begin{cases} C_{y-1}^2 (0.5)^2 = \frac{2}{(y-1)!(3-y)!}(0.5)^2, & y = 1,2,3 \\ 0, & otherwise \end{cases}$$

$E(Y|X=0) = 0 \times C_0^2 (0.5)^2 + 1 \times C_1^2 (0.5)^2 + 2 \times C_2^2 (0.5)^2 = 1$

$$E\left(Y \middle| X=1\right)=1\times C_0^2\left(0.5\right)^2+2\times C_1^2\left(0.5\right)^2+3\times C_2^2\left(0.5\right)^2=2$$

(c)

$$E\left(X\right)=p=0.5,\qquad \mathrm{var}\left(X\right)=p\left(1-p\right)=0.25$$

$$E\left(Y\right)=np=1.5,\qquad \mathrm{var}\left(Y\right)=np\left(1-p\right)=0.75$$

$$E\left(XY\right)=0.5\times 0\times E\left(Y \middle| X=0\right)+0.5\times 1\times E\left(Y \middle| X=1\right)=1$$

$$\mathrm{cov}\left(X,Y\right)=E\left(XY\right)-E\left(X\right)E\left(Y\right)=1-0.5\times 1.5=0.25$$

$$\rho=\frac{\mathrm{cov}\left(X,Y\right)}{\sqrt{\mathrm{var}\left(X\right)}\sqrt{\mathrm{var}\left(Y\right)}}=\frac{0.25}{\sqrt{0.25}\sqrt{0.75}}=\frac{1}{\sqrt{3}}$$

3.61（93-淡江-企管）

2. 一袋中 5 個紅球，3 個白球，甲每次由袋中拿出一個球，試問在下列兩種情況下，甲至少要幾次才能保證拿到白球的機率高過 95%？
(a)拿出球看完顏色後放回。(5%)
(b)拿出球看完顏色後不放回。(5%)

【解】

(a)

X 爲超幾何分配：$N=8$ 、$S=3$ 、n 、$P\left(X\right)=\dfrac{C_x^S C_{n-x}^{N-X}}{C_n^N}=\dfrac{C_x^3 C_{n-x}^5}{C_n^5}$

$$P\left(X\ge 1\right)>95\% \quad\Rightarrow\quad P\left(X=0\right)<5\%$$

$$\Rightarrow\quad \frac{C_0^3 C_n^5}{C_n^8}=\frac{\left(8-n\right)!5!}{8!\left(5-n\right)!}=\frac{\left(8-n\right)\left(7-n\right)\left(6-n\right)}{8\times 7\times 6}<5\%$$

$$\Rightarrow\quad \left(8-n\right)\left(7-n\right)\left(6-n\right)<16.8$$

$$\Rightarrow\quad n\ge 5$$

(b)

X 爲二項分配：n 、$p=\dfrac{3}{8}$ 、$P\left(X\right)=C_x^n p\left(1-p\right)^{n-x}=\dfrac{n!}{x!\left(n-x!\right)}\left(\dfrac{3}{8}\right)^x\left(\dfrac{5}{8}\right)^{n-x}$

$$P\left(X\ge 1\right)>95\% \quad\Rightarrow\quad P\left(X=0\right)<5\%$$

$$\Rightarrow\quad \left(\frac{3}{8}\right)\left(\frac{5}{8}\right)^{n-1}<5\%$$

$$\Rightarrow\quad \left(n-1\right)\ln\left(\frac{5}{8}\right)>\ln\left(\frac{8\times 5\%}{3}\right)$$

$$\Rightarrow\quad n>5.28 \quad\Rightarrow\quad n\ge 6$$

3.62（93-淡江-企管）

7. Let X be a random variable with density

$$f(x)=\begin{cases}0, & x<0\\ 0.5, & 0\le x\le 1\\ qe^{-\alpha x}, & x>1\end{cases}$$

(a) Find q, assuming that α is known. (5%)

(b) Find the hazard $h(t)$. $h(t)=f(t)/[1-F(t)]$ (10%)

【解】

(a)

$$\int f(x)dx=1 \quad\Rightarrow\quad \int_0^1 0.5dx+\int_1^\infty qe^{-\alpha x}dx=0.5+\frac{q}{\alpha}e^{-\alpha}=1 \quad\Rightarrow\quad q=\frac{\alpha e^{\alpha}}{2}$$

(b)

$$F(t)=P(X\le t)=\begin{cases}0, & t<0\\ \frac{1}{2}t, & 0\le t\le 1\\ 1-\frac{1}{2}e^{-\alpha(t-1)}, & t>1\end{cases}$$

$$h(t)=\frac{f(t)}{1-F(t)}=\begin{cases}0, & t<0\\ \dfrac{0.5}{1-\frac{1}{2}t}=\dfrac{1}{2-t}, & 0\le t\le 1\\ \dfrac{\frac{\alpha}{2}e^{-\alpha(t-1)}}{\frac{1}{2}e^{-\alpha(t-1)}}=\alpha, & t>1\end{cases}$$

3.63（93-政大-國貿）

1. Find the values of the constant K that define the following density functions for two discrete random variables and for one continuous random variable.

(i) $f(y)=K2^{-y},\ y=1,2,\cdots$. （5%）

(ii) $f(y)=K2^{-y}/y,\ y=1,2,\cdots$. （5%）

(iii) $f(y)=K(y-y^2)$, where $0\le\alpha<y<\beta\le 1$ and $K>0$. (5%)

【解】

(1)

$$\sum_{y=1}^{\infty} K2^{-y} = K\sum_{y=1}^{\infty}\left(\tfrac{1}{2}\right)^{y} = K\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \quad \Rightarrow \quad K = 1$$

(2)

令 $q = \sum_{y=0}^{\infty} p^{y} = \dfrac{1}{1-p}, \quad 0 < p < 1$

兩邊對 p 積分：$\int q dp = \sum_{y=0}^{\infty}\dfrac{1}{y+1}p^{y+1} = \int \dfrac{1}{1-p}dp = -\ln(1-p) \quad \Rightarrow \quad \sum_{y=1}^{\infty}\dfrac{1}{y}p^{y} = -\ln(1-p)$

$$\sum_{y=1}^{\infty} K\frac{2^{-y}}{y} = K\sum_{y=1}^{\infty}\frac{1}{y}\left(\tfrac{1}{2}\right)^{y} = 1 \quad \Rightarrow \quad -K\ln\left(\tfrac{1}{2}\right) = 1 \quad \Rightarrow \quad K = \ln 2 = 0.6931$$

(3)

$$\int_0^1 K\left(y-y^2\right)dy = K\left(\frac{1}{2}-\frac{1}{3}\right) = 1 \quad \Rightarrow \quad K = 6$$

3.64（93-政大-國貿）

> 2. Let Z have a Poisson density with parameter λ. Utilize Chebyshev's inequality to verify the following inequalities:
>
> (i) $P\left(Z \le \dfrac{\lambda}{2}\right) \le \dfrac{4}{\lambda}$. (5%)
>
> (ii) $P\left(Z \ge 2\lambda\right) \le \dfrac{1}{\lambda}$. (5%)

【解】

Chebyshev 定理：$P\left(\left|Z-\mu\right| \le k\sigma\right) \ge 1-\left(\dfrac{1}{k}\right)^2$ 或 $P\left(\left|Z-\mu\right| \ge k\sigma\right) \le \left(\dfrac{1}{k}\right)^2$

Z 為參數 λ 之卜瓦松分配，則 $\mu = E(Z) = \lambda$ 、 $\sigma = \sqrt{\operatorname{var}(Z)} = \sqrt{\lambda}$

(1)

$$\frac{\lambda}{2} = \lambda - k\sqrt{\lambda} \quad \Rightarrow \quad k = \frac{\sqrt{\lambda}}{2}$$

$$P\left(Z \le \frac{\lambda}{2}\right) \le P\left(\left|Z-\mu\right| \ge \frac{\sqrt{\lambda}}{2}\sigma\right) \le \left(\frac{2}{\sqrt{\lambda}}\right)^2 = \frac{4}{\lambda}$$

(2)

$$2\lambda = \lambda - k\sqrt{\lambda} \quad \Rightarrow \quad k = -\sqrt{\lambda}$$

$$P\left(Z \ge 2\lambda\right) \le P\left(\left|Z-\mu\right| \ge \sqrt{\lambda}\sigma\right) \le \left(\frac{1}{\sqrt{\lambda}}\right)^2 = \frac{1}{\lambda}$$

3.65（93-政大-財管）

8. (10%) Much is made of the fact that certain mutual funds outperform the market year after year (this is, the return from holding shares in the mutual fund is higher than the return from holding a portfolio such as the S&P 500). For concreteness, consider a ten-year period and let the population be the 4,170 mutual funds reported in *The Wall Street Journal* on January 1, 1995. By saying that performance relative to the market is random, we mean that each fund has a 50-50 chance of outperforming the market in any year and that performance is independent from year to year.
   a. If performance relative to the market is truly random, what is the probability that any particular fund outperforms the market in all 10 years?

【解】

X 為二項分配：$n=10$ 、 $p=0.5$ 、 $P(X)=C_x^n p^x (1-p)^{n-x} = \dfrac{10!}{x!(10-x)!}(0.5)^{10}$

$P(x=0)=0.5^{10}=\dfrac{1}{1024}$

3.66（93-政大-財管）

7. (10%) Let X be a random variable distributed as Normal(5, 4). Find the probabilities of the following events:
   a. P(X > 6).
   b. P(|X − 5| > 1)

【解】

X 為常態分配：$\mu=5$ 、 $\sigma=\sqrt{4}=2$

(a)

$$P(X>6)=P\left(z=\frac{X-5}{2}>\frac{6-5}{2}=0.5\right)=P(z>0.5)$$

(b)

$$P(|X-5|>1)=P\left(|z|=\left|\frac{X-5}{2}\right|>\frac{1}{2}\right)=P(|z|>0.5)$$

3.67（93-政大-風管）

1. Consider the discrete random variable X with probability function f(x)

x	f(x)
-1	0.5
0	0.3
1	0.1
3	0.1

a. Determine the cumulative distribution function of X. (3%)

b. Determine the Moment Generating Function of X. (5%)

【解】

(a)

$$F(X)=P(X\le x)=\begin{cases}0, & x<-1\\ 0.5, & -1\le x<0\\ 0.8, & 0\le x<1\\ 0.9, & 1\le x<3\\ 1, & x\ge 3\end{cases}$$

(b)

$$m(t)=E\left(e^{xt}\right)=0.5\times(-1)\times e^{-t}+0.1\times 1\times e^{t}+0.1\times 3\times e^{3t}=-0.5e^{-t}+0.1e^{t}+0.3e^{3t}$$

3.68（93-政大-風管）

2. Consider the continuous random variable X with probability density function

$f(x)=x/18,\ 0\le x\le 6$.

a. Calculate $E[X^3]$. (3%)

b. Determine the median of X. (5%)

【解】

(a)

$$E\left(X^3\right)=\int_0^6 x^3\frac{x}{18}dx=\frac{1}{18}\times\frac{x^5}{5}\bigg|_0^6=\frac{432}{5}=86.4$$

(b)

令 m_e 爲中位數，則

$$\int_0^{m_e} f(x)dx=0.5\ \Rightarrow\ \int_0^{m_e}\frac{x}{18}dx=\frac{x^2}{36}\bigg|_0^{m_e}=\frac{m_e^2}{36}=0.5\ \Rightarrow\ m_e=\sqrt{36\times 0.5}=4.24$$

3.69（93-政大-風管）

3. Let X be normally distributed with mean 10 and standard deviation 5. Compute the following.
   a. $P(0 \le X < a) = 0.9$. Find a. (5%)
   b. $P((x-10)^2 \le b) = 0.99$. Find b. (5%)

【解】

X 為常態分配：$\mu = 10$ 、$\sigma = 5$

(a)

$P(0 \le X < a) = P(-2 \le z < a') = 0.9 \quad \Rightarrow \quad a' = 1.4242$ （查表）

$a' = \dfrac{a-10}{5} = 1.4242 \quad \Rightarrow \quad a = 17.1210$

(b)

$P\left((x-10)^2 \le b\right) = P\left(|x-10| \le \sqrt{b}\right) = P\left(|z| \le \dfrac{\sqrt{b}}{5}\right) = 0.99$

$\Rightarrow \quad \dfrac{\sqrt{b}}{5} = 2.58$（查表） $\Rightarrow \quad b = 3.5917$

3.70（93-政大-風管）

4. An expert sharpshooter misses the target, on the average, 5% of the time.
   a. What is the probability that in 20 shots, the sharpshooter will hit the target 18 or more times? (5%)
   b. What is the probability that the sharpshooter will miss the target for the second time on the 25^{th} shot? (3%)

【解】

(a)

X 為二項分配：$n = 20$ 、 $p = 0.95$ 、 $P(X) = C_x^n p^x (1-p)^{n-x}, \quad x = 1, 2, \ldots, n$

$P(X \ge 18) = P(X = 18) + P(X = 19) + P(X = 20)$

$= \dfrac{20 \times 19}{2} \times 0.95^{18} \times 0.05^2 + 20 \times 0.95^{19} \times 0.05^1 + 0.95^{20} = 0.9245$

(b)

第 25 發需失誤，而且前 24 發恰有 1 發失誤：

$P = (1-p) \times C_1^{24} p^{23} (1-p)^1 = 24 \times 0.95^{23} \times 0.05^2 = 0.0184$

3.71（93-政大-風管）

5. The number of weekly breakdowns of a computer is a Poisson distribution with mean equal to 2. a. What is the probability that the computer breaks down 5 times in one week? (3%) b. Suppose that the computer breaks down at time T = 0. What is the probability that the next breakdown will occur after 1 week? (5%)

【解】

X 為卜瓦松分配：$\lambda = 2$（次/週）、$P(X) = \dfrac{\lambda^x e^{-\lambda}}{x!} = \dfrac{2^x e^{-2}}{x!}$

(a)

$$P(X=5) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{2^5 e^{-2}}{5!} = 0.0036$$

(b)

$$P(X=0) = e^{-2} = 0.1353$$

$$P(T>1) = 1 - P(X=0) = 1 - 0.1353 = 0.8647$$

3.72（93-政大-風管）

7. Evaluate the following integrals. a. $\int_0^\infty 5x^5 e^{-x} dx$. (7%) b. $\int_0^1 2x(1-x)^2 dx$. (7%)

【解】

(a)

$$\begin{aligned}\int_0^\infty 5x^5 e^{-x} dx &= -5x^5 e^{-x}\Big|_0^\infty + \int_0^\infty 5\times 5x^4 e^{-x} dx \\ &= -5\times 5x^4 e^{-x}\Big|_0^\infty + \int_0^\infty 5\times 5\times 4x^3 e^{-x} dx \\ &\vdots \\ &= \int_0^\infty 5\times 5\times 4\times 3\times 2\times 1 e^{-x} dx \\ &= 600\end{aligned}$$

(b)

$$\int_0^1 2x(1-x)^2 dx = \int_0^1 \left(2x - 4x^2 + 2x^3\right) dx = \frac{2}{2} - \frac{4}{3} + \frac{2}{4} = \frac{1}{6}$$

3.73（93-政大-風管）

8. If the exponent of e of a bivariate normal density is

$$\frac{-1}{102}[(x+2)^2-2.8(x+2)(y-1)+4(y-1)^2].$$

Find

a. $\mu_1, \mu_2, \sigma_1, \sigma_2$, and ρ. (7%)

b. $\mu_{Y|x}$ and $\sigma^2_{Y|x}$. (7%)

【解】

$$f(x,y)=\frac{1}{2\pi\sqrt{|\boldsymbol{\Sigma}|}}e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}(\mathbf{x}-\boldsymbol{\mu})},\quad \boldsymbol{\Sigma}=\begin{bmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{bmatrix},\quad (\mathbf{x}-\boldsymbol{\mu})=\begin{bmatrix}x-\mu_x\\ y-\mu_y\end{bmatrix}$$

$$-\frac{1}{102}\left[(x+2)^2-2.8(x+2)(y-1)+4(y-1)^2\right]=-\frac{1}{2}\frac{1}{51}[x+2,y-1]\begin{bmatrix}1 & -1.4\\ -1.4 & 2\end{bmatrix}\begin{bmatrix}x+2\\ y-1\end{bmatrix}$$

$$\Rightarrow\quad \boldsymbol{\Sigma}^{-1}=\frac{1}{51}\begin{bmatrix}1 & -1.4\\ -1.4 & 4\end{bmatrix}\quad\Rightarrow\quad \boldsymbol{\Sigma}=\begin{bmatrix}\sigma_x^2 & \sigma_{x,y}\\ \sigma_{x,y} & \sigma_y^2\end{bmatrix}=\begin{bmatrix}100 & 35\\ 35 & 25\end{bmatrix}$$

(a)

$$\mu_x=-2,\quad \mu_y=1,\quad \sigma_x=\sqrt{100}=10,\quad \sigma_y=\sqrt{25}=5,\quad \rho=\frac{\sigma_{x,y}}{\sigma_x\sigma_y}=\frac{35}{10\times 5}=0.7$$

(b)

$$\mu_{Y|x}=\mu_Y+\frac{\sigma_{X,Y}}{\sigma_X^2}(x-\mu_X)=1+\frac{35}{100}\times(x-2)=0.3+0.35x$$

$$\sigma^2_{Y|x}=\sigma_Y^2-\frac{\sigma_{X,Y}\sigma_{X,Y}}{\sigma_X^2}=25-\frac{35\times 35}{100}=12.75$$

3.74（93-政大-風管）

9. Verify the identity $\sum_{i=1}^{n}(X_i-\mu)^2=\sum_{i=1}^{n}(X_i-\overline{X})^2+n(\overline{X}-u)^2$. (10%)

【解】

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\mu)^2&=\sum_{i=1}^{n}(X_i-\overline{X}+\overline{X}-\mu)^2\\ &=\sum_{i=1}^{n}\left[(X_i-\overline{X})^2-2(X_i-\overline{X})+(\overline{X}-\mu)^2\right]\\ &=\sum_{i=1}^{n}(X_i-\overline{X})^2-2(\overline{X}-\mu)\sum_{i=1}^{n}(X_i-\overline{X})+\sum_{i=1}^{n}(\overline{X}-\mu)^2\\ &=\sum_{i=1}^{n}(X_i-\overline{X})^2+n(\overline{X}-\mu)^2\end{aligned}$$

其中，$\sum_{i=1}^{n}\left(X_i-\overline{X}\right)=\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}\overline{X}=0$

3.75（93-台大-商學）

12. Let X and Y have the joint p.m.f.

$f(x,y)=\dfrac{x+2y}{18}$, x=1,2, y=1,2,

The marginal probability mass function are, respectively,

$f_1(x)=\dfrac{2x+6}{18}$, x=1,2 and $f_2(y)=\dfrac{3+4y}{18}$, y=1,2

Find the correlation coefficient of X and Y ?

(A) -0.45 (B) -0.025 (C) 0.45 (D) 0.21 (E) None of the above

【解】

$$E(X)=1\times\frac{2+6}{18}+2\times\frac{4+6}{18}=\frac{28}{18}$$

$$E(X^2)=1^2\times\frac{2+6}{18}+2^2\times\frac{4+6}{18}=\frac{48}{18}$$

$$E(Y)=1\times\frac{3+4}{18}+2\times\frac{3+8}{18}=\frac{29}{18}$$

$$E(Y^2)=1^2\times\frac{3+4}{18}+2^2\times\frac{3+8}{18}=\frac{51}{18}$$

$$E(XY)=(1\times1)\times\frac{1+2}{18}+(2\times1)\times\frac{2+2}{18}+(1\times2)\times\frac{1+4}{18}+(2\times2)\times\frac{2+4}{18}=\frac{45}{18}$$

$$\rho=\frac{E(XY)-E(X)E(Y)}{\sqrt{E(X^2)-E(X)E(X)}\sqrt{E(Y^2)-E(Y)E(Y)}}$$

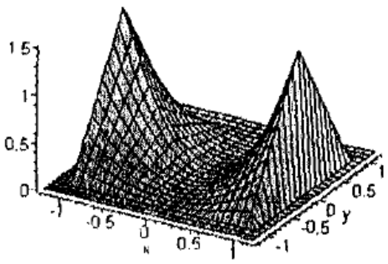
$$=\frac{45-\dfrac{28\times29}{18}}{\sqrt{48-\dfrac{28\times28}{18}}\sqrt{51-\dfrac{29\times29}{18}}}=-0.0255$$

3.76（93-台大-商學）

17. Let X and Y have the joint p.d.f.

$f(x,y)=3/2x^2(1-|y|)$, $-1<x,1,-1<y<1$,

The graph of $Z=f(x,y)$ is given in the below figure.



Let A={(x,y):0<x<1,o<y<x}. The probability that (X,Y) falls in A is given by

(A)8/23 (B)5/13 (C)9/40 (D)3/20 (E) None of the above.

【解】

$$f(x,y)=\frac{3}{2}x^2(1-|y|)=\begin{cases}\frac{3}{2}x^2(1+y), & -1<x<1,-1<y<0\\ \frac{3}{2}x^2(1-y), & -1<x<1,0\le y<1\\ 0, & otherwise\end{cases}$$

$$P(A)=\int_0^1\int_0^x\frac{3}{2}x^2(1-y)dydx=\int_0^1\frac{3}{2}x^2\left(x-\frac{1}{2}x^2\right)dx=\frac{3}{2}\times\frac{1}{4}-\frac{3}{4}\times\frac{1}{5}=\frac{9}{40}$$

3.77（93-台大-商學）

18. Given the moment generating function (m.g.f.) of X is

$M(t)=\dfrac{e^t/2}{1-e^t/2}$, $t<\ln 2$.

then the p.m.f. of X is

(A) $f(x)=(\frac{1}{2})^x$, x=1,2,3.....

(B) $f(x)=(\frac{3}{2})^x$, x=1,2,3.....

(C) $f(x)(\frac{2}{3})^x$, x=1,2,3.....

(D) $f(x)(-\frac{2}{3})^x$, x=1,2,3.....

【解】

$$m(t)=E\left(e^{xt}\right)=\sum_{x=1}^{\infty}e^{xt}p^x=\sum_{x=1}^{\infty}\left(e^tp\right)^x=\frac{e^tp}{1-e^tp},\quad \text{when } e^tp<1$$

$\Rightarrow \quad p=\dfrac{1}{2}$

3.78（93-台大-商學）

> 19. Let $Y_1<Y_2<Y_3\ldots<Y_{13}$ be the order statistics associated with 13 independent observations of a random sample from a continuous-type distribution with 35^{th} percentile $\pi_{0.35}$. Please find $P(Y_3<\pi_{0.35}<Y_7)$=?
>
> (A) 0.3402 (B) 0.4327 (C) 0.6132 (D) 0.7573 (E) None of the above

【解】

X 為二項分配：$n=13$ 、 $p=0.35$ 、 $P(X)=\dfrac{13!}{x!(13-x)!}(0.35)^x(0.65)^{13-x}$

$$\begin{aligned}P(3<X<7)&=P(X=4)+P(X=5)+P(X=6)\\&=0.2222+0.2154+0.1546\\&=0.5922\end{aligned}$$